Dynamic Decision Making: Fully Probabilistic Design

Miroslav Kárný et al

school@utia.cas.cz, Adaptive Systems Department, ÚTIA AVČR

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List of Contributors

- Kamil Dedecius, dedecius@utia.cas.cz
- Ivan Nagy, nagy@utia.cas.cz
- Lenka Pavelková, pavelkov@utia.cas.cz
- Evgenia Suzdaleva, suzdalev@utia.cas.cz
- Ondřej Tichý, otichy@utia.cas.cz

Domain and Approach

Topic

This text provides a unified basis of dynamic decision making under uncertainty & incomplete knowledge. It designs & applies strategy↰ that

- converts available knowledge↰ into an optional action↰,
- concerns a system↰, i.e. a part of the World,
- faces consistently an inevitable uncertainty↰,
- respects decision-maker's constraint↰s,
- meets decision-maker's aim↰s as best as possible.

Research and Application Domains

The topic description indicates an extreme width of the addressed problems. Consequently, it deals with an extreme range of

- research and application domains covering control engineering [Ast70], artificial intelligence [San99], pattern recognition [Rip97], economics [Sta00], social sciences [Arr95];
- methodologies and techniques like statistical decision making [Wal50, Sav54, DeG70, Ber85], fuzzy decision making [Tri00], domain-specific solutions [SB01], transition of statistical physics technique to economical domain [RM00];
- synonyms and notation variations, for instance, action vs. decision vs. input; output vs. response.

What Is Specific?

The adopted solution falls within Bayesian decision-making paradigm [Wal50, DeG70, Ber85]. The text is specific by its

- stress on dynamic decision making requiring design↰ of strategies generating sequences of actions;
- systematic use of probabilistic description to all basic DM elements↰;
- top down presentation stressing a common logical structure in solving rather diverse problems;
- constructive, problem-driven, approach;
- vocabulary combining terms from various domains [KBG+06].

Introduction

This part

- provides practical examples of decision making (DM↰) that serve as an informal introduction into the general problem addressed;
- characterises the thought audience & acknowledgements.

Examples of DM

Example 1 (On Enrolling at this Course)

aim	*to learn something interesting, to get credits*
system	*teacher, school mates, the personal future*
action	$\{$*enter, not-enter*$\}$ *this course*
knowledge	*syllabus, gossip of older students*
ignorance	*true content of the course*
uncertainty	*degree of simplicity, intellectual state of the teacher, personal ability to perceive*
constraint	*spent time, brain effort, schedule*
dynamics	*one-shot decision with long-term consequences like lost time, usefulness in future life...*

Examples of DM

Example 2 (Estimation of Table Length)

aim	*to provide an information serving for the table displacement using either a small lift or staircase*
system	*the table and space around it*
action	*the upper estimate of the table length*
knowledge	*a personal guess, available observations*
ignorance	*the true length of the table*
uncertainty	*measurement errors*
constraint	*spent time, the tape precision*
dynamics	*one-shot decision with longer term consequences like the lost time and energy on measurements or a trial table displacement...*

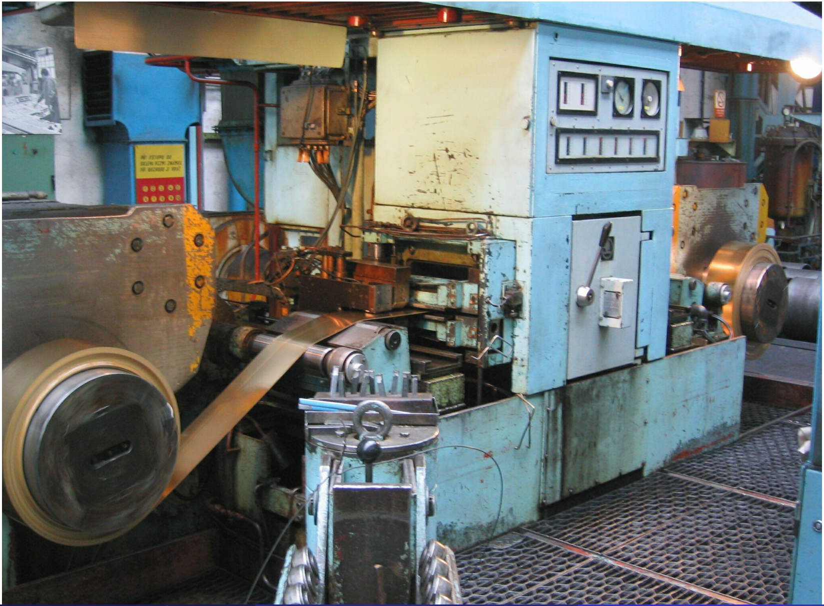
Examples of DM

Example 3 (Control of Metal Thickness)

aim↰: *to get the metal of a constant thickness*

Rolled Metal

System: Rolling Mill



Actions: Rolling Force, Rolling Speed, Rolling Tensions, . . . ($\sim$ 300 options)

Other DM Elements

knowledge	observations – input-output thickness, speeds, tensions ... (40 channels) – and personal knowledge (at least 6 months' learning)
ignorance	detailed properties of the mill and of the rolled material (the mill more hammers than rolls)
uncertainty	measurement errors, eccentricity of rolls, responses of actuators to commands, mill aging, ...
constraint	the applied forces and tensions and their changes, control period (about 10 ms), precision of the thickness- and pressure-measuring sensors
dynamics	time-delay between the measured input thickness and applied rolling force (more than 20 control periods), dynamics of actuators.

Example 4 (Control of the Traffic in Town)

aim ↰ *to exploit fully the available capacity of town roads, for instance, to minimise the average travelling time*

System: a Traffic Region in Town

Action: Transformation of the System

. . . to the System

Figure: Radical solution of the traffic problem

… or Varying Traffic Lights and Signs

Other DM Elements

knowledge↰	off-line statistical data, observation↰s of the traffic intensity & road occupancy, visual inspections, Figure 7
ignorance↰	the car flow evolving over the space & time, queue lengths
uncertainty↰	measurement errors, un-measured quantities, e.g., the number of parking cars, weather, congestions, behaviour of drivers,. . .
constraint↰	the available capacity of the transportation system, priorities of public transportation, safety regulations, information systems, complexity of evaluations, . . .
dynamics↰	the traffic is a random spatially-distributed process, light changes at single cross-road have far reaching influence, recall "green wave" and its violation, consequences of even a minor accident, priorities of state-guests,. . .

Audience and Supporters

- In spite of the (mis)used mathematics, this is not mathematical text.
- A basic course of mathematical analysis suffices for understanding of the explanation logic.
- On the other hand, the text touches quite deep formulation concepts so that the proper audience consists of specialising Master and PhD students as well as researchers who are interested in criticism, development and non-trivial applications of decision making theory.
- The text reflects decades of the first author work and as well as of the explicitly listed contributors. To name all people and institutions who influenced the text would be extremely long and boring for audience. Current and former colleagues know that their work is appreciated. This allows us to name only grants that supported this version, MŠMT 1M0572, GA ČR 102/08/0567.

Basic Notions

On Notions, Notations and Conventions

- This part summarises basic notions and notations used throughout.
- The conventions listed here are mostly followed in this work. If an exception is necessary it is introduced at the place of its validity.
- The respective notions are introduced within the text when they are used for first time. They are emphasised.
- Jumps to majority of definitions are possible in the PDFLaTeX version. Thus, a reader can scan this part in a rather shallow way.
- The presentation starts with general conventions. The core of this part provides briefly characterised basic notions. Then, the used vocabulary is commented.

Notions and Notations

- *pd* probability density, f, means Radon-Nikodým derivative of a probabilistic measure [Rao87b].
  The argument name determines meaning of the pd.
- *mappings* are marked by sf fonts.
- *expectation* is denoted E or E_{f} to stress the pd f used.
- *set* $X^\star$ denotes the range of X.
- *subset* $X_\star$ is a part of $X^\star$.
- *cardinality* $|X^\star|$ denotes the number of members in the set $X^\star$.
- *vector length* ℓ_X means the number of entries in the vector X.
- *defining equality* $\equiv$ is the equality by definition.
- *timed quantity* X_t is a quantity X at the discrete time instant labelled by $t \in t^\star \equiv \{1, \ldots, h\}$.
- *horizon* $h \leq \infty$ concerns decision, prediction, control . . .
- *time index* $X_{t;i}$ is an ith entry of the array X at time t.

The semicolon in the subscript stresses that the first index is time.

Notions and Notations

- *sequence* $X^{k:l}$ denotes the sequence $(X_i)_{i=k}^{l}$.
  $X^{k:l}$ is empty sequence adding nothing to prior knowledge↰ if $l < k$.
  $X^t \equiv X^{1:t} \equiv (X_i)_{i=1}^{t}$ is the sequence from the time moment 1 till t.
- *support* $\mathrm{supp}\,[\,\mathsf{f}(X)]$ is the subset $X_\star$ of $X^\star$ on which $\mathsf{f}(X) > 0$.
- *quantity* is a multivariate (measurable) mapping. Its detailed description is mostly unimportant.
  This notion corresponds with random variable used in probability theory, [Ren72]. The adopted name stresses that probability serves us as a DM↰ tool and not as a primary object. It also stresses our inclination to deal with a numerical description of physical entities.
- *realisation* is a quantity value for a fixed argument.
  Often, the quantity↰ and its realisation↰ are not distinguished. The context implies the proper alternative.

Interface of the Theory to Reality

Agreement 1 (Connections to Reality)

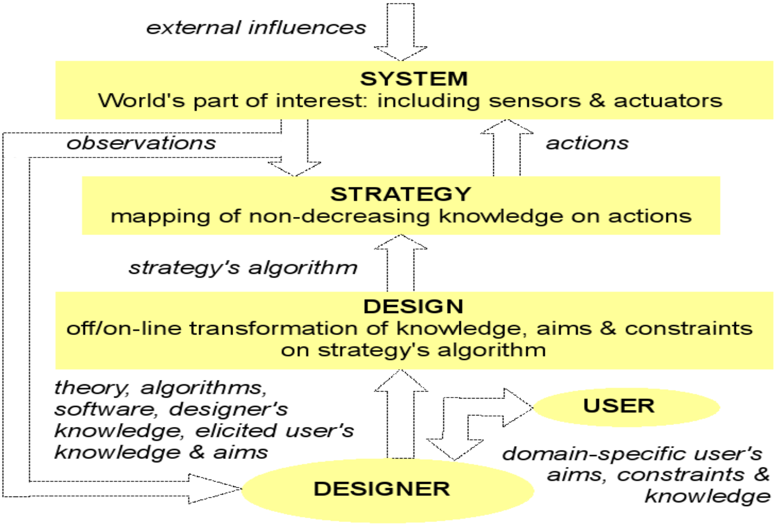
Physical connections of DM elements to the real world

- *sensors,*
- *transmission lines,*
- *actuators,*
- *. . .*

are taken here as a part of the physical system dealt with.

All considered quantities and mappings are mathematical entities living in an abstract calculating machine.

Abstraction of DM Problem



DM Concerns Interactions

In DM, at least two entities interacts: system↰ and decision maker↰.

- *system* is a part of the World that is of interest to a decision maker↰ who should either describe or influence it. arxsimul

  The system↰ is specified with respect to the aim↰ of the decision maker↰ and with respect to its available tools. In other words, the penetrable boundaries of the system are implied by the decision task.
- *decision maker* is a person or mechanism who has to select and apply action↰s.
  To avoid gender offences a decision maker is referred by it.
  A compound decision maker↰ is possible.
- The presented normative theory should help the decision maker↰ to select the proper, from its view-point, action↰ among alternatives.

Action, Behaviour and Aim

- *action* $\equiv$ decision, $A \in A^{\star}$, is the value of a quantity that can be directly chosen by the decision maker for reaching its aim.
  Terms "action" and "decision" are taken as synonyms.
  A decision task arises iff there are several actions available, $|A^{\star}| > 1$.
  The action is selected with the intention to reach a specific aim as closely as possible.
- *behaviour* , $B \in B^{\star}$, consists of realisations of all quantities considered by the decision maker in the addressed decision-making task within the time span determined by the horizon of interest.
- *aim* specifies the desired behaviour of the closed decision loop formed by the decision strategy and system.

Strategy, Decision Rule and Feedback

The described theory intends to support any (rational) decision maker↰ in its choice of appropriate sequence of action↰s for any foreseen realisation↰ of the behaviour↰. Thus, it has to design strategy↰.

- *strategy* $\equiv$ decision strategy S is a sequence of mappings from the behaviour set $B^\star$ to the action set $A^\star$, $\mathsf{S} \equiv (\mathsf{S}_t : B^\star \to A^\star_t)_{t \in t^\star}$.
- *decision rule* S_t is a mapping $\mathsf{S}_t : B^\star \to A^\star_t$ that assigns the action↰ $A_t \in {A_t}^\star$ to the behaviour↰ $B \in B^\star$ at time t.
- *feedback* means that the strategy↰ maps behaviours on actions, which generally influence the behaviour.

Knowledge and Ignorance

An applicable strategy↰ can process only the knowledge↰ available:

- *knowledge* $\equiv$ decision knowledge $\mathcal{K}_{A^\star} \in \mathcal{K}_{A^\star}{}^\star$ is the part of the behaviour $B \in B^\star$ available to the decision maker↰ for the choice of the action↰ $A \in A^\star$. The abbreviation $\mathcal{K}_{t-1} = \mathcal{K}_{A_t^\star}$ is used.
  Time shift stresses the inevitable time-delay in gathering knowledge and possibility to use it for DM↰.
  For example, if just data values $D \in D^\star$ are available for constructing an estimate $\hat{\Theta}$ of an unknown quantity↰ $\Theta \in \Theta^\star$, then the knowledge↰ is $\mathcal{K}_{\hat{\Theta}^\star} \equiv D$. Often, the knowledge↰ includes the observed past.
- *ignorance* $\equiv$ decision ignorance $\mathcal{G}_{A^\star} \in \mathcal{G}_{A^\star}^\star$ is the part of the behaviour $B \in B^\star$ unavailable to the decision maker↰ for the choice of the action ↰ $A \in A^\star$. The abbreviation $\mathcal{G}_t = \mathcal{G}_{A_t^\star}$ is used.
  An estimated quantity Θ belongs to the ignorance↰ $\mathcal{G}_{\hat{\Theta}}$ of the estimate $\hat{\Theta}$.
  Often, ignorance↰ contains yet unobserved future.

Knowledge Accumulation

Action $A_t \in A_t{}^\star$ splits behaviour B into ignorance $\mathcal{G}_t$ & knowledge $\mathcal{K}_{t-1}$

$$\text{behavior} = B = (\mathcal{G}_t, A_t, \mathcal{K}_{t-1}) = (\text{ignorance, action, knowledge}). \tag{1}$$

A single realisation B splits differently with respect to actions $A_t \in A_t{}^\star$, $A_\tau \in A_\tau{}^\star$ with different knowledge $\mathcal{K}_{t-1} \neq \mathcal{K}_{\tau-1}$ and, consequently, different ignorance $\mathcal{G}_t \neq \mathcal{G}_\tau$.

$$B = (\mathcal{G}_t, A_t, \mathcal{K}_{t-1}) = (\mathcal{G}_\tau, A_\tau, \mathcal{K}_{\tau-1}).$$

An accumulation of the knowledge and reduction of the ignorance via sequential observations in the increasing time $t \in t^\star$ formalises the notion

- *observation* $\Delta_t \in \Delta_t{}^\star$ consists of quantities included in the ignorance $\mathcal{G}_t$ of A_t and in the knowledge $\mathcal{K}_t$ of A_{t+1}.

  Often, $\Delta_t = Y_t =$ the system output at time t.

Informationally Causal Decision Rules and Strategies

The applicable decision rules have to be informationally causal.

- *causal decision rule* S_t is a mapping that assigns the action $A_t \in A_t{}^\star$ to its knowledge $\mathcal{K}_{t-1} \in \mathcal{K}_{t-1}{}^\star$.
  The action A_t generated by a causal decision rule S_t is uninfluenced by the related ignorance $\mathcal{G}_t$.
- *estimator* is a causal decision rule $\mathsf{S}_t : \mathcal{K}^\star_{t-1} \to \hat{X}^\star_t$ that assigns an estimate $\hat{\Theta}_t$ of an unknown quantity $\Theta_t \in \Theta_t{}^\star$ to the knowledge $\mathcal{K}_{t-1}$.
- *causal strategy* $\mathsf{S} \equiv (\mathsf{S}_t : \mathcal{K}_{t-1} \to A^\star_t)_{t \in t^\star}$ is a sequence made of causal decision rules.

We deal with causal decision rules and causal strategies only. Thus, the term "causal" can mostly be dropped.

Our Central Topic Is Design of DM Strategy

- *design* selects a decision rule↰ or a strategy↰.
- *static design* selects a single decision rule↰.
- *dynamic design* chooses a strategy↰.
- *dynamics* means any circumstance that calls for the dynamic design↰.
- *designer* is a person (or a group) who makes the strategy selection. Authors and readers of this text are supposed to be designers and the term "we" used within the text is to be read: we designers.
  The designers work for the users whose aims should be reached by using the strategy↰ designed.
  Decision makers, designers and users are mostly identified in this text.

Engineers Solve DM in Its Entirety

- *DM* $\equiv$ decision making means the design↰ and the use of a decision rule↰ or a strategy↰.

The quest for applicability forces us to select

- *admissible strategy* $\mathsf{S} \equiv (\mathsf{S}_t)_{t \in t^\star}$, which
  - is causal, i.e. $\mathsf{S} \equiv (\mathsf{S}_t)_{t \in t^\star} \equiv (\mathsf{S}_t : \mathcal{K}^\star_{t-1} \to A^\star_t)_{t \in t^\star}$
  - meets a given constraint↰.
- *constraint* is any circumstance restricting the set of strategies $\mathsf{S}^\star$ among which the designer↰ can choose.
- *physical constraints* limit actions $(A^\star_t)_{t \in t^\star}$.
- *informational constraints* determine knowledge↰ $(\mathcal{K}_{t-1})_{t \in t^\star}$ available for selection of actions $(A_t)_{t \in t^\star}$ and ignorance↰ $(\mathcal{G}_t)_{t \in t^\star}$ considered but unavailable for the action choice.

Practically Admissible Strategies

Practically, constraints on complexity of the strategy↰ have to be respected.

- *practically admissible strategy* is an admissible strategy↰ that respects constraint↰ limiting the complexity of the DM.

The complexity is considered with respect to the computational resources (computational time and memory used) available at the design and application stages.

- The majority of discussed problems in which the complexity constraints play a role are computationally hard in terms of computer sciences [Gol08]. An intuitive understanding of the computational complexity suffices to us.

Key Obstacle: DM Faces Uncertainty

- *uncertainty* occurs if the strategy↰ S does not determine uniquely the behaviour↰ $B \in B^\star$. Then, for a fixed S, there is a bijective mapping
$$\mathsf{W}(\mathsf{S},\cdot): N^\star \to B^\star. \tag{2}$$
The argument $N \in N^\star$ is called uncertainty.
For DM↰, uncertainties that for a fixed strategy↰ S lead to the same behaviour↰ are equivalent. This allows us to consider bijective $\mathsf{W}(\mathsf{S},\cdot)$ only.
- *uncertain behaviour* arises if the uncertainty set $N^\star \neq \emptyset$.
The DM↰ uncertainty is delimited by fixing the system↰, the decision maker↰ and the set of admissible strategies $\mathsf{S}^\star$.
- Uncertainty is permanently a part of the ignorance↰.
- An unknown quantity↰ $\Theta \in \Theta^\star$ in behaviour↰ makes it uncertain in the operationally same way as an external unobserved noise.
- Uncertainty covers incomplete knowledge, vague preferences of the decision maker↰, randomness, etc.

Comments on Vocabulary

The used vocabulary has various counter-parts, for instance,

- *input* $\equiv$ system input, $U \in U_t^\star$, is an action, which is supposed to influence ignorance $\mathcal{G}_t$.

A manipulated valve position influencing a fluid flow is the input.
An estimate $\hat{\Theta}$ of an unknown quantity Θ is an action that is not the input. The estimate describes the system but has no influence on it.

- *output* $\equiv$ system output $Y \in Y^\star$ is an observable quantity that informs the decision maker about the behaviour. To be or not to be output or input is relative. A pressure measured in a heated system is an output. A pressure applied to a system is an input.

- *controller* is a strategy assigning the input U_t to knowledge $\mathcal{K}_{t-1}$. The proportional controller with a proportionality constant C is a causal control strategy $(\mathcal{K}_{t-1}^\star \equiv Y_{t-1}^\star \to U_t^\star : \; U_t = -CY_{t-1})_{t \in t^\star}$.

Formalisation of DM Under Uncertainty

Quest for Normative Theory of DM Under Uncertainty

- This part summaries the design principle we exploit in solving decision-making tasks.
- Recall: DM theory should help the decision maker↰ to opt for an action↰. The option concerns either a description of a system↰ or an influence on it.
- The presentation describes a general way how to understand and face uncertainty↰ that causes incomplete ordering of strategies even when preferential ordering↰ of possible behaviours is complete.
- The complete ordering↰ of strategies, harmonised with the preferential ordering↰ of behaviours, is then proposed.

Ordering of Behaviours

A DM design makes sense if the decision maker prefers some behaviours.

- *preferential ordering* of the decision maker is ordering $\preccurlyeq_{B^\star}$ of behaviours $B \in B^\star$.
  It is the relation $\preccurlyeq_{B^\star}$ on pairs $\left({}^aB, {}^bB\right) \in B^\star \times B^\star$

$$ {}^aB \preccurlyeq_{B^\star} {}^bB \text{ reads } {}^aB \text{ is preferred against } {}^bB. \tag{3}$$

- The desirable consistency of preferences restricts it to be transitive

$$\left({}^aB \preccurlyeq_{B^\star} {}^bB \wedge {}^bB \preccurlyeq_{B^\star} {}^cB\right) \Rightarrow {}^aB \preccurlyeq_{B^\star} {}^cB. \tag{4}$$

Completion of Behaviour Ordering

- The preferential ordering $\preceq_{B^\star}$ is generally a partial ordering as the decision maker is often unable or unwilling to compare all behaviours. We shall counteract it by employing
- *preferential quantity* , which is a part of ignorance introduced with the aim to get complete ordering of behaviours containing it.

  This non-standard consideration allows us to assume that $\preceq_{B^\star}$ is
- *complete ordering* of the behaviour set $B^\star$ $\preceq_{B^\star}$ that is able to compare preferentially any ${}^{a}B,\ {}^{b}B \in B^\star$: either ${}^{a}B \preceq_{B^\star} {}^{b}B$ or ${}^{b}B \preceq_{B^\star} {}^{a}B$. The ordering $\preceq_{B^\star}$ induces the strict ordering $\prec_{B^\star}$ and the preferential equivalence $\approx_{B^\star}$

$$\begin{aligned} {}^{a}B \prec_{B^\star} {}^{b}B \quad &\Leftrightarrow \quad {}^{a}B \preceq_{B^\star} {}^{b}B \wedge \neg({}^{b}B \preceq_{B^\star} {}^{a}B) \\ {}^{a}B \approx_{B^\star} {}^{b}B \quad &\Leftrightarrow \quad {}^{a}B \preceq_{B^\star} {}^{b}B \wedge {}^{b}B \preceq_{B^\star} {}^{a}B \end{aligned} \tag{5}$$

Numerical Representation of Ordering

- The algorithmic solution of DM↰ tasks, we aim at, is enabled by constructing numerical representation of the preferential ordering↰ $\preceq_{B^\star}$ by a real-valued loss↰ Z.
- *loss* $\mathsf{Z} : B^\star \to (-\infty, \infty)$ quantifies the degree of the aim achievement if it is strictly isotonic with the preferential ordering↰, i.e. ${}^{a}B \prec_{B^\star} {}^{b}B \Leftrightarrow \mathsf{Z}({}^{a}B) < \mathsf{Z}({}^{b}B)$ and ${}^{a}B \approx_{B^\star} {}^{b}B \Leftrightarrow \mathsf{Z}({}^{a}B) = \mathsf{Z}({}^{b}B)$.
- The loss↰ measures a posteriori the quality of each realisation↰ B. The smaller is the loss↰ value the better.

Existence of Numerical Representation

The extended real line $(-\infty, \infty)$ with the ordinary (complete) strict ordering $<$ has the topology generated by open intervals [Bou66]. It has a countable $<$-dense subset of rational numbers: for any real pair $a < b$ there is c in the $<$-dense subset such that $a < c < b$. It is intuitively clear that the loss↰ may exist if $(B^\star, \preceq)$ has a countable $\prec_{B^\star}$-dense subset. Proof of the following proposition can be found in [Deb54] or [Fis70].

Proposition 1 (Existence of the Loss)

If a countable $\prec_{B^\star}$-dense set in $(B^\star, \preceq_{B^\star})$ exists then there is loss↰ Z representing $\preceq_{B^\star}$

$$
{}^aB \prec_{B^\star} {}^bB \Leftrightarrow \mathsf{Z}\left({}^aB\right) < \mathsf{Z}\left({}^bB\right) \ \wedge
$$
$$
{}^aB \approx_{B^\star} {}^bB \Leftrightarrow \mathsf{Z}\left({}^aB\right) = \mathsf{Z}\left({}^bB\right).
$$

Non-Uniqueness of Loss

- The loss $Z(\cdot)$ representing a preferential ordering $\preccurlyeq_{B^{\star}}$ is not unique.
- The freedom in selection of the loss can be restricted in a meaningful way, for instance, by requiring it to be continuous with respect to the $\preccurlyeq_{B^{\star}}$-topology, e.g., [Fis70].
- There is a danger that uniqueness of the loss is obtained at too high price: unnecessary additional assumptions may exclude meaningful completions of preferences supplied by the user.

Complete Ordering of Strategies

DM↰ design consists of selecting the "best" strategy↰ ${}^{O}\mathsf{S}$ among

- *compared strategies* form a subset $\mathsf{S}_{\star} \subset \mathsf{S}^{\star}$ of admissible strategies consisting of at least two strategies.

It means that there is a complete ordering↰ of compared strategies $\preccurlyeq_{\mathsf{S}_{\star}}$, which can be viewed as restriction of a complete ordering of all admissible strategies from $\mathsf{S}^{\star}$

- *ordering of strategies* $\preccurlyeq_{\mathsf{S}^{\star}}$ is interpreted

$$ {}^{a}\mathsf{S} \preccurlyeq_{\mathsf{S}^{\star}} {}^{b}\mathsf{S} \Leftrightarrow {}^{a}\mathsf{S} \text{ is better than } {}^{a}\mathsf{S}. \tag{6} $$

- $\preccurlyeq_{\mathsf{S}^{\star}}$ has to be harmonised with the preferential ordering↰ $\preccurlyeq_{B^{\star}}$.

Towards a Prescriptive DM Theory

- The choice of the complete ordering of strategies has ambition to be prescriptive, as objective as possible. Thus, it has to be applicable to any preferential ordering numerically represented by the corresponding loss. It has to suit to all DM tasks differing in $S_\star$ of compared strategies, which are subsets of the set of admissible strategies $S^\star$.
- The configuration of the decision maker – system, determining behaviour and its finer structure, is assumed to be fixed.
- The complete ordering of strategies $\preccurlyeq_{S^\star}$, will be represented by the "expected" loss T.
- The quotation marks at expectation are used temporarily. They serve the discussion, which shows that, under widely acceptable conditions, it has to be mathematical expectation of utility.

The Design Principle

- The "expected" loss represents preferences among strategies. This implies the design principle:
- *optimal design* selects an admissible strategy that leads to the smallest value of the "expected" loss.
- *optimal strategy* is a minimiser ${}^{O}\mathsf{S}$ of the "expected" loss.

Towards "Expected" Loss

We start with a preferential ordering $\preceq_{B^\star}$ represented by a fixed loss Z.

- Substitution of the mapping (2), relating the strategy and the uncertainty to the behaviour, into the loss Z converts the loss into a function $Z_S(N)$ of the strategy S and uncertainty N

$$Z_S(N) \equiv Z(W(S, N)), \; S \in S^\star, \; N \in N^\star. \tag{7}$$

- Various strategies generate the set $Z_{S^\star}$ of functions of uncertainties

$$Z_{S^\star} \equiv \{Z_S : \; N^\star \to (-\infty, \infty), \; Z_S(N) \equiv Z(W(S, N))\}_{S \in S^\star}. \tag{8}$$

Ordering of Strategies Orders Possible Loss Realisations

- In quest for objectivity, we deal with a complete ordering↰ of all admissible strategies

$$ {}^{a}\mathsf{S} \preccurlyeq_{\mathsf{S}^\star} {}^{b}\mathsf{S} \Leftrightarrow {}^{a}\mathsf{S} \text{ is preferred againts } {}^{b}\mathsf{S},\ \ {}^{a}\mathsf{S},\ {}^{b}\mathsf{S} \in \mathsf{S}^\star. \tag{9} $$

- The complete ordering of strategies induces the complete ordering↰ of functions from the set $\mathsf{Z}_{\mathsf{S}^\star}$ (8)

$$ \mathsf{Z}_{{}^{a}\mathsf{S}} \preccurlyeq_{\mathsf{Z}_{\mathsf{S}^\star}} \mathsf{Z}_{{}^{b}\mathsf{S}} \Leftrightarrow {}^{a}\mathsf{S} \preccurlyeq_{\mathsf{S}^\star} {}^{b}\mathsf{S}. \tag{10} $$

- We assume that a numerical representation of $\preccurlyeq_{\mathsf{Z}_{\mathsf{S}^\star}}$ exists, Proposition 1, i.e. a mapping $\mathsf{T} : \mathsf{Z}_{\mathsf{S}^\star} \to (-\infty, \infty)$ exists

$$ \mathsf{Z}_{{}^{a}\mathsf{S}} \preccurlyeq_{\mathsf{Z}_{\mathsf{S}^\star}} \mathsf{Z}_{{}^{b}\mathsf{S}} \Leftrightarrow \mathsf{T}(\mathsf{Z}_{{}^{a}\mathsf{S}}) \le \mathsf{T}(\mathsf{Z}_{{}^{b}\mathsf{S}}). \tag{11} $$

- The equivalence (10) provides the numerical representation of the strategy ordering $\preccurlyeq_{\mathsf{S}^\star}$ via the functional T (11)

$$ {}^{a}\mathsf{S} \preccurlyeq_{\mathsf{S}^\star} {}^{b}\mathsf{S} \Leftrightarrow \mathsf{T}(\mathsf{Z}_{{}^{a}\mathsf{S}}) \le \mathsf{T}(\mathsf{Z}_{{}^{b}\mathsf{S}}). \tag{12} $$

Integral Representation of “Expectation”

- The functional T, numerically representing ordering of strategies $\preccurlyeq_{\mathsf{S}^\star}$ via (12), is an “expectation” T of the loss↰ $\mathsf{Z}_\mathsf{S}(N) = \mathsf{Z}(\mathsf{W}(\mathsf{S}, N))$. It is required to be universally applicable to any loss↰ $\mathsf{Z}(B)$, i.e. to any preferential ordering↰ of behaviours $\preccurlyeq_{B^\star}$. The “expectation” for a specific preferential ordering↰ is then taken as the restriction of the found T on the set (8) generated by a specific loss↰ and by compared strategies↰.
- We express the functional T in an integral form. For it, we adopt rather technical conditions. Essentially, the applicability to any smooth loss↰ and a local version of “linearity” of T are required.

Towards Integral Representation of "Expectation"

Requirement 1 (The "Expectation" Domain)

The "expectation" T *acts on the union* $Z^{\star}_{S^{\star}}$ *of the sets* $Z_{S^{\star}}$ *(8) of functions with a common uncertainty set* $N^{\star}$

$$Z^{\star}_{S^{\star}} \equiv \cup_{Z \in Z^{\star}} Z_{S^{\star}}. \tag{13}$$

The set $Z^{\star}_{S^{\star}}$ *is required to contain a subset of*

- *test losses , which are zero out of a compact subset* $\emptyset \neq N_{\star}$ *of* $N^{\star}$ *and continuous on* $N_{\star}$*, where supremum norm defines the corresponding topology allowing a meaningful definition of continuity.*

The "expectation" is assumed to be a sequentially continuous, and uniformly continuous functional on the test losses. It is, moreover, additive on losses with non-overlapping supports

$$T[Z_1 + Z_2] = T[Z_1] + T[Z_2] \text{ if } Z_1 Z_2 = 0,\ Z_1,\ Z_2 \in Z^{\star}_{S^{\star}}.$$

Integral Representation of "Expectation"

Technical Requirement 1 allows us to get an integral representation of the "expectation" searched for. More exact formulation and proof of the corresponding theorem as well as definitions of the adopted non-common terms can be found in [Rao87b] (Theorem 5, Chapter 9).

Proposition 2 (Integral "Expectation" Form)

Under Requirement 1, the "expectation" T *of* $\mathsf{Z} \in \mathsf{S}^{\star}_{\mathsf{S}^{\star}}$ *reads*

$$\mathsf{T}[\mathsf{Z}] = \int_{N^{\star}} \mathsf{U}(\mathsf{Z}(N), N)\, \mu(dN), \tag{14}$$

specified by a finite regular nonnegative Borel measure μ *and by utility:*

- *utility is the mapping* U *in (14). It satisfies* $\mathsf{U}(0, N) = 0$. *It is continuous in values of* $\mathsf{Z}(\cdot)$*, almost everywhere (a.e.) on* $N_{\star}$*, bounded a.e. on* $N_{\star}$ *for each* Z *in the set of test losses.*

- The test losses↰ are widely applicable and their consideration represents no practical restriction. The "expectation" is applicable even out of this set of functions.
- The continuity requirements on T are also widely acceptable.
- The linearity of T on functions with non-overlapping support seems to be sound. Indeed, any loss↰ $\mathsf{Z} \in \mathsf{Z}^{\star}_{\mathsf{S}^{\star}}$ can be written as $\mathsf{Z} = \mathsf{Z}\chi_{\omega} + \mathsf{Z}(1 - \chi_{\omega}) \equiv \mathsf{Z}_1 + \mathsf{Z}_2$, $\mathsf{Z}_1\mathsf{Z}_2 = 0$ with χ_{ω} denoting an indicator of a set $\omega \subset N_{\star} \subset N^{\star}$. The indicator χ_{ω} is a smooth function that equals 1 within ω and it is zero outside of it.

  The loss↰ "expected" on the set ω and its complement should sum to the loss "expected" on the whole set of arguments.

Universal Character of Utility U and Measure μ

- The utility U that shapes the original loss allows the decision maker to express the attitude toward design consequences and their risks: the decision maker might be risk aware, prone, or indifferent [KR78].
- The utility U and the nonnegative measure μ are universal for the whole set of test functions. U and μ are (almost) "objective", i.e. suitable for a range of decision tasks facing the same uncertainty.

Ordering of Strategies vs. Preferential Behaviour Ordering

- The introduced ordering of strategies $\preccurlyeq_{S^\star}$ has been up to now unrelated to the preferential ordering $\preccurlyeq_{B^\star}$ of behaviours $B \in B^\star$.

What makes them reasonably harmonised?

- We have to avoid undoubtedly bad orderings of strategies, which select strategy leading surely to bad behaviour as the optimal one.
  - We are not aware another, generally applicable, harmonisation requirement!
  - We specify below what bad strategies mean.

Dominance Ordering

- The set $\mathsf{Z}^{\star}_{\mathsf{S}^{\star}}$ (13) of functions of uncertainty can be equipped with the partial ordering

$$ {}^{a}\mathsf{Z}(N) \leq {}^{b}\mathsf{Z}(N), \ \forall N \in N^{\star}. \tag{15} $$

- The partial ordering (15) restricted on the subset $\mathsf{Z}_{\mathsf{S}^{\star}}$ (8) of the set $\mathsf{Z}^{\star}_{\mathsf{S}^{\star}}$ (given by a fixed loss Z) induces the partial dominance ordering $\preccurlyeq^{d}_{\mathsf{S}^{\star}}$ of strategies $\mathsf{S} \in \mathsf{S}^{\star}$
- *dominance ordering* means that the strategy ${}^{b}\mathsf{S}$ is dominated by the strategy ${}^{a}\mathsf{S}$,

$$ {}^{a}\mathsf{S} \preccurlyeq^{d}_{\mathsf{S}^{\star}} {}^{b}\mathsf{S} \text{ iff } \mathsf{Z}_{{}^{a}\mathsf{S}}(N) \leq \mathsf{Z}_{{}^{b}\mathsf{S}}(N), \quad \forall N \in N^{\star}. \tag{16} $$

The strategy ${}^{b}\mathsf{S}$ is strictly dominated by the strategy ${}^{a}\mathsf{S}$, ${}^{a}\mathsf{S} \prec^{d}_{\mathsf{S}^{\star}} {}^{b}\mathsf{S}$ if the inequality (16) is sharp on a $N_{\star} \subset N^{\star}$ with $\mu(N_{\star}) > 0$, see (14).

Dominated Strategies Are Bad

The optimal design depends on the "expected" loss $\mathsf{T}[\mathsf{Z}_\mathsf{S}]$ chosen. Its choice has to guarantee that unequivocally bad strategy must not be chosen as the optimal strategy.

Definitely, a strictly dominated strategy is accepted as bad one as it leads to a higher loss than another admissible strategy irrespectively of the uncertainty realisation.

Requirement 2 (Quest for Non-Dominance)

The expected loss must be chosen so that the optimal design performed on any nontrivial (comparison allowing) subset $\mathsf{S}_\star$ of compared strategies (a subset of admissible strategies $\mathsf{S}^\star$) must not take a strictly dominated strategy as the optimal strategy.

Utility Must Be Isotonic

Proposition 3 (Isotonic "Expectation")

Assume that there is a strategy in $\mathsf{S}^\star$ for which the "expected" loss is finite. Then, Requirement 2 is fulfilled iff the "expectation" is strictly isotonic with the strict dominance ordering of strategies $\prec^d_{\mathsf{S}^\star}$.

Proof:

a) We prove by contradiction that strict isotonicity guarantees non-dominance. Let $\mathsf{T}[\mathsf{Z}]$ be strictly isotonic on its domain $\mathsf{Z}^\star_{\mathsf{S}^\star}$ (8) and ${}^{o}\mathsf{S} \in \mathsf{S}^\star$ be a minimiser of the "expected" loss. The minimiser gives necessarily a finite value of the corresponding $\mathsf{T}[\mathsf{Z}_{\,{}^{o}\mathsf{S}}]$. Let ${}^{d}\mathsf{S} \in \mathsf{S}^\star$ dominate strictly ${}^{o}\mathsf{S}$. Then, because of the construction of ${}^{o}\mathsf{S}$ via minimisation, the strict dominance and the strictly isotonic nature of T, we get the following contradictory inequality

$$\mathsf{T}[\mathsf{Z}_{\,{}^{o}\mathsf{S}}(N)] \underbrace{\leq}_{\text{minimum}} \mathsf{T}[\mathsf{Z}_{\,{}^{d}\mathsf{S}}(N)] \underbrace{<}_{\text{strictly isotonic}} \mathsf{T}[\mathsf{Z}_{\,{}^{o}\mathsf{S}}(N)] < \infty.$$

Proof Continued

b) We prove by contradiction that the use of an "expectation" T that is not strictly isotonic leads to the violation of Requirement 2. If $\mathsf{T}[\mathsf{Z}_\mathsf{S}]$ is not strictly isotonic on its domain $\mathsf{Z}_{\mathsf{S}^\star}$ (8) then there is a strategy ${}^{a}\mathsf{S} \in \mathsf{S}^\star$ strictly dominated by the strategy ${}^{d}\mathsf{S} \in \mathsf{S}^\star$ such that

$$\mathsf{T}[\mathsf{Z}_{{}^{d}\mathsf{S}}] \geq \mathsf{T}[\mathsf{Z}_{{}^{a}\mathsf{S}}].$$

If we restrict the set of strategies $\mathsf{S}^\star$ to the pair $\mathsf{S}_\star \equiv \{{}^{d}\mathsf{S}, {}^{a}\mathsf{S}\}$ then ${}^{a}\mathsf{S}$ can always be taken as the optimal strategy. Thus, Requirement 2 is not met with such "expectation" T.

Proposition 4 (Utility Must Be Increasing In Loss)

The optimal design avoids dominated strategies iff the utility in the "expectation" (14) is increasing in its first argument.

Proof Non-negativity of the measure μ (14) makes the claim obvious. □

Uncertainties in DM Have Random Structure

- The "expectation" scaled by any positive factor preserves the ordering, which represents. Thus, the measure μ (14) can be normalised to probabilistic measure. This choice preserves the constant utilities $\mathsf{T}[\mathit{constant}] = \mathit{constant}$.
- Back-substitution $B = \mathsf{W}(\mathsf{S}, N)$ (2) into (14) gives

$$
\begin{aligned}
{}^{a}\mathsf{S} \preccurlyeq_{\mathsf{S}^\star} {}^{b}\mathsf{S} &\Leftrightarrow \mathsf{T}_{{}^{a}\mathsf{S}}(\mathsf{Z}) \leq \mathsf{T}_{{}^{b}\mathsf{S}}(\mathsf{Z}), \quad {}^{a}\mathsf{S}, {}^{b}\mathsf{S} \in \mathsf{S}^\star \\
\mathsf{T}_{\mathsf{S}}(\mathsf{Z}) &= \int_{B^\star} \mathsf{U}\left(\mathsf{Z}(B), \mathsf{W}^{-1}(\mathsf{S}, B)\right) \mu_{\mathsf{S}}(\mathrm{d}B) \qquad (17)
\end{aligned}
$$

where the non-negative probabilistic measure μ_{S} is image of the measure μ under the mapping $\mathsf{W}(\mathsf{S}, \cdot)$ (2).

Let us Get Rid Off Measures and Utilities

The involved measures $\mu_{\mathsf{S}},\ \mathsf{S} \in \mathsf{S}^\star$ are assumed to have

- *Radon–Nikodým derivative* (probability density pd↰) $\mathsf{f}_{\mathsf{S}}(B)$ is defined with respect to a product dominating measure denoted $\mathrm{d}B$, [Rao87b].

In the treated cases, $\mathrm{d}B$ is either Lebèsgue or counting measure.

- expectation↰ is then

$$\mathsf{E}_{\mathsf{S}}[\mathsf{I}_{\mathsf{S}}] = \int_{B^\star} \mathsf{I}_{\mathsf{S}}(B)\mathsf{f}_{\mathsf{S}}(B)\mathrm{d}B = \mathsf{T}_{\mathsf{S}}(\mathsf{Z}). \tag{18}$$

  and it is determined by

- *closed loop model* , which is the pd $\mathsf{f}_{\mathsf{S}}(B)$

  The expectation E_{S} is applied to a strategy-dependent

- *performance index*

$$\mathsf{I}_{\mathsf{S}}(B) = \mathsf{U}(\mathsf{Z}(B), \mathsf{W}^{-1}(\mathsf{S}, B)). \tag{19}$$

- *traditional design* considers (19) with a performance index $\mathsf{I} = \mathsf{I}_{\mathsf{S}}$ independent of the optimised strategy $\mathsf{S} \in \mathsf{S}^\star$.
- *objective expectation* is the expectation (18) that serves to all DM↰ tasks with a common uncertainty.
- *objective pd* is the pd specifying the objective expectation.

A Few Comments

Remark 1

- *Dependence of the performance index↰ on strategy↰ arises through the non-standard second argument of utility↰. This dependence does not occur in the traditional design↰. Its consideration is postponed to Section 16, where it helps us to justify the fully probabilistic design.*
- *The existence of pds is unnecessary but it helps us to deal with simpler objects.*
- *Mostly, the character of the dominating measure is unimportant and we stick predominantly to the Lebèsgue-type notation.*
- *According to our agreements on simplified notation, the pds* $\mathsf{f}(N)$ *and* $\mathsf{f}(B)$ *are different functions.*

We Have Arrived to the Point Where Books on DM Start

- *formalised DM design* (20)

$$\begin{array}{llcl}
\text{optimal strategy} & {}^{o}\mathsf{S} & \in & \mathrm{Arg}\min_{\mathsf{S}\in\mathsf{S}^\star} \mathsf{E}[\mathsf{I}_\mathsf{S}] \text{ with} \\
\text{admissible strategy} & \mathsf{S} & \equiv & (\mathsf{S}_t : \mathcal{K}_{t-1} \to A_t{}^\star)_{t\in t^\star} \\
\text{performance index} & \mathsf{I}_\mathsf{S} & : & B^\star \to (-\infty,\infty) \text{ evaluating} \\
\text{behaviour} & B & \equiv & (\mathcal{G}_t, A_t, \mathcal{K}_{t-1}) \\
 & & \equiv & (\mathrm{ignorance}, \mathrm{action}, \mathrm{knowledge}) \\
\text{expectation} & \mathsf{E}_\mathsf{S}[\mathsf{I}_\mathsf{S}] & \equiv & \int_{B^\star} \mathsf{I}_\mathsf{S}(B)\mathsf{f}_\mathsf{S}(B)\,\mathrm{d}B \\
\text{closed loop model} & \mathsf{f}_\mathsf{S} & : & B^\star \to [0,\infty],\ \int_{B^\star} \mathsf{f}_\mathsf{S}(B)\,\mathrm{d}B = 1.
\end{array}$$

- *traditional DM design* coincides with the formalised DM design for a strategy-independent performance index $\mathsf{I} = \mathsf{I}_\mathsf{S}$.

The Formalisation Has THE Gap

We aim at the optimal design↰ but we can perform only

- *practically optimal design* selects a practically admissible strategy↰ giving the smallest value of the expected performance index↰ while respecting limited computational resources during the design.
- The optimal design↰ adapts simply to the choice of a strategy↰ of a pre-specified complexity. It suffices to optimise over them.
- Operational formal tools for practically optimal design↰ are unavailable. It is not known how to make the optimal design↰ of a pre-specified complexity.

  Essentially, the lack of the answer to simple questions like: How to find the best proportional controller made with 10 algebraic operations available? is the main barrier of the applicability of the theory describing the optimal design↰.

... Global Plan of Further Explanations

- Auxiliary tools needed for solution of the optimisation (20) are prepared, Sections 8, 9 and 10.
- The general solution of the traditional DM design↰ of the optimal strategy↰ is found, Section 11. It reveals the need for learning. Its solution is summarised in Section 13.
- Asymptotic properties of the design and learning are inspected in Sections 12, 15.
- General, fully probabilistic design (FPD↰) allowing dependence of the performance index↰ on strategy is introduced in Section 16 and solved in Section 18. Among others, it provides tools for a realistic construction of DM elements↰ used in design. This construction forms an independent part of this text, which is finely structured before its start.

Decisive Optimisation Tools

- This part provides basic tools serving us for solving DM tasks.
- Section 8 recalls elementary calculus with pd↰s.
- Section 9 summarises properties of conditional expectation we need.
- The design relies of basic DM lemma, presented in Section 10. The lemma also exemplifies the elements on which the design operates.

Calculus with Pds

The pd ↰ is the main technical tool we deal with. Here, the joint pd f of $B \equiv (\alpha, \beta, \gamma) \in B^\star$ is connected to related pds.

- *joint pd* $\mathsf{f}(\alpha, \beta|\gamma)$ of α, β conditioned on γ is the pd on $(\alpha, \beta)^\star$ projecting the joint pd $\mathsf{f}(B) \equiv \mathsf{f}(\alpha, \beta, \gamma)$ on the cross-section of $B^\star$ given by a fixed γ.
- *marginal pd* $\mathsf{f}(\alpha|\gamma)$ of α conditioned on γ is the pd on $\alpha^\star$ projecting $\mathsf{f}(B) \equiv \mathsf{f}(\alpha, \beta, \gamma)$ on the cross-section of $B^\star$ given by a fixed γ while no information on β is available.
- *unconditional pd* is formally obtained if just trivial condition is considered. Then, the conditioning symbol $|$ is dropped .
  The pd $\mathsf{f}(\alpha, \beta)$ is the joint pd ↰ in a lower dimension. It is marginal pd ↰ of the pd $\mathsf{f}(\alpha, \beta, \gamma)$ similarly as $\mathsf{f}(\beta)$.
- *conditionally independent* quantities α and β, under the condition γ, meet
$$\mathsf{f}(\alpha, \beta|\gamma) = \mathsf{f}(\alpha|\gamma)\mathsf{f}(\beta|\gamma) \Leftrightarrow \mathsf{f}(\alpha|\beta, \gamma) = \mathsf{f}(\alpha|\gamma). \tag{21}$$

Basic Operations with Pds

Proposition 5 (Calculus with Pds)

For any $B \equiv (\alpha, \beta, \gamma) \in B^\star$*, the following relations hold*

- *non-negativity means that all variants of pds are non-negative.*
- *normalisation means that all variants of pds have unit integral over the domain of quantities before conditioning sign* $|$*.*
- *chain rule for pds holds* $\mathsf{f}(\alpha, \beta|\gamma) = \mathsf{f}(\alpha|\beta, \gamma)\mathsf{f}(\beta|\gamma)$*.*
- *marginalisation means* $\mathsf{f}(\beta|\gamma) = \int_{\alpha^\star} \mathsf{f}(\alpha, \beta|\gamma)\, \mathrm{d}\alpha$*.*
- *Bayes rule*

$$\mathsf{f}(\beta|\alpha, \gamma) = \frac{\mathsf{f}(\alpha|\beta, \gamma)\mathsf{f}(\beta|\gamma)}{\mathsf{f}(\alpha|\gamma)} = \frac{\mathsf{f}(\alpha|\beta, \gamma)\mathsf{f}(\beta|\gamma)}{\int_{\beta^\star} \mathsf{f}(\alpha|\beta, \gamma)\mathsf{f}(\beta|\gamma)\, \mathrm{d}\beta}$$
$$\propto \mathsf{f}(\alpha|\beta, \gamma)\mathsf{f}(\beta|\gamma). \tag{22}$$

- *proportionality* $\propto$ *is equality with an implicit presence of a unique normalisation-determined factor independent of the pd's argument before the conditioning sign* $|$*.*

Proof For motivation see [Pet81], a more precise and more technical treatment exploits the measure theory [Rao87b]. An intermediate insight can be gained by considering a loss↰ dependent only on a part of B or with some parts of B "fixed by the condition", [KHB+85]. □

Remark 2

The Bayes rule (22) is a simple consequence of previous formulas. Its importance in this text cannot be exaggerated, cf. Propositions 14, 15.

Pds of Transformed Quantities

Often, the pd $\mathsf{f}_{\Upsilon}(\beta)$ of a multivariate image $\beta = \Upsilon(\alpha)$ of the quantity α with a given pd $\mathsf{f}(\alpha)$ is needed.

Proposition 6 (Pds of Transformed Quantities)

The expectation $\mathsf{E}_{\Upsilon}[\mathsf{I}(\beta)] = \int_{\beta^\star} \mathsf{I}(\beta)\mathsf{f}_{\Upsilon}(\beta)\,\mathrm{d}\beta$*, acting on functions* $\mathsf{I}(\beta) : \beta^\star \to (-\infty, \infty)$*, coincides with the expectation*

$$\begin{aligned} \mathsf{E}[\mathsf{I}(\beta)] &= \int_{\alpha^\star} \mathsf{I}(\Upsilon(\alpha))\mathsf{f}(\alpha)\,\mathrm{d}\alpha \text{ iff} \\ \int_{\Upsilon(\alpha_\star)} \mathsf{f}_{\Upsilon}(\Upsilon(\alpha))\,\mathrm{d}\Upsilon(\alpha) &= \int_{\alpha_\star} \mathsf{f}(\alpha)\,\mathrm{d}\alpha, \end{aligned}$$

for all measurable subsets $\alpha_\star \subset \alpha^\star$*.*

Proof It follows from the possibility to approximate any measurable function by piece-wise constants. □

Pds of Smoothly Transformed Quantities

Proposition 7 (Pds of Smoothly Transformed Quantities)

Let α *be a real vector ,* $\alpha \equiv [\alpha_1, \ldots, \alpha_{\ell_\alpha}]$ *and* $\Upsilon = [\Upsilon_1, \ldots, \Upsilon_{\ell_\alpha}]$ *bijection with finite continuous partial derivatives almost everywhere on* $\alpha^\star$

$$J_{ij}(\alpha) \equiv \frac{\partial \Upsilon_i(\alpha)}{\partial \alpha_j}, \; i, j = 1, \ldots, \ell_\alpha, \tag{23}$$

for all entries Υ_i *of* Υ *and entries* α_j *of* α*.*
Then,

$$\mathsf{f}_\Upsilon(\Upsilon(\alpha))|J(\alpha)| = \mathsf{f}(\alpha), \quad \textit{where} \tag{24}$$

$|\cdot|$ *is absolute value of the argument determinant.*

Proof Proposition describes substitutions in multivariate integrals; see, for instance, [Rao87b, Jar84]. □

- It is useful to summarise basic properties of expectation. They simplify formal manipulations.
- For this text, it is sufficient to take the expectation in a naive way as an integral weighted by the conditional pd $\mathsf{f}(\cdot|\gamma)$. The textbook [Rao87b] can be consulted for a rigourous treatment.
- Properties are formulated for the conditional expectation. The unconditional case is obtained by omitting the condition.
- Note that whenever the expectation is applied to an array function V it should be understood as the array of expectations $[\mathsf{E}(V)]_i \equiv \mathsf{E}(V_i)$.
- Elegant manipulations with expectations are sometimes dangerous as the used pd is not obvious from the notation. Sometimes, explicit use of arguments over which expectation is taken helps. In explanatorily critical cases, integral expressions are used.
- The symbol E_f is used to stress the expectation-defining pd.

Useful Properties of Expectation

Proposition 8 (Basic Properties of E)

For arbitrary real functions ${}^{a}\mathsf{I}(B)$, ${}^{b}\mathsf{I}(B)$, $B \in B^{\star}$ *on which the conditional expectation* $\mathsf{E}[\cdot|\gamma]$ *is well defined,* $\mathsf{E}[\cdot|\gamma]$ *has the following properties.*

- *expectation linearity*

$$\mathsf{E}[\alpha(\gamma)\,{}^{a}\mathsf{I} + \beta(\gamma)\,{}^{b}\mathsf{I}|\gamma] = \alpha(\gamma)\mathsf{E}[\,{}^{a}\mathsf{I}|\gamma] + \beta(\gamma)\mathsf{E}[\,{}^{b}\mathsf{I}|\gamma]$$

  for arbitrary real coefficients α, β *depending at most on* γ.

- *chain rule for expectation*

$$\mathsf{E}\left[\mathsf{E}[\cdot|\gamma,\zeta]|\gamma\right] = \mathsf{E}[\cdot|\gamma] \tag{25}$$

  for an arbitrary additional condition ζ.

Proof The definition and integral expression provides the results. $\square$

Proposition 9 (Moments and Jensen Inequality)

- *conditional covariance of a vector* α
  $\mathrm{cov}[\alpha|\gamma] \equiv \mathsf{E}\left[(\alpha - \mathsf{E}[\alpha|\gamma])(\alpha - \mathsf{E}[\alpha|\gamma])'|\gamma\right]$ *is related to the non-central moments through the formula*

$$\mathrm{cov}[\alpha|\gamma] = \mathsf{E}[\alpha\alpha'|\gamma] - \mathsf{E}[\alpha|\gamma]\mathsf{E}[\alpha'|\gamma], \quad \text{' is transposition.} \tag{26}$$

- *Jensen inequality bounds expectation of a convex function*
  $\mathsf{I}_\gamma : \alpha^\star \to (-\infty, \infty)$

$$\mathsf{E}[\mathsf{I}_\gamma(\alpha)|\gamma] \geq \mathsf{I}_\gamma\left(\mathsf{E}[\alpha|\gamma]\right). \tag{27}$$

Proof All statements can be verified by using the integral expression of the expectation. Proof of the Jensen inequality can be found, e.g., in [Vaj82]. □

Basic DM Lemma

The construction of the optimal strategy↰ solving the traditional DM design↰ (20) relies on the key proposition that reduces the minimisation over mappings to an "ordinary" minimisation. It is formulated for the static design↰ selecting a single decision rule↰.

Proposition 10 (Basic DM↰ Lemma of Traditional Design)

The optimal admissible decision rule ${}^{O}\mathsf{S}$ *solving the traditional DM design↰ (20) can be chosen as deterministic one* ${}^{O}\mathsf{S}(\mathcal{K}_{A^\star}) \equiv {}^{O}A(\mathcal{K}_{A^\star})$*. It can be constructed value-wise as follows. To each* $\mathcal{K}_{A^\star} \in \mathcal{K}_{A^\star}{}^{\star}$,

$$
{}^{O}A(\mathcal{K}_{A^\star}) \in \mathrm{Arg} \min_{A \in A^\star} \mathsf{E}[\mathsf{I}|A, \mathcal{K}_{A^\star}] \tag{28}
$$

provides the value of the optimal decision rule↰ corresponding to the considered argument $\mathcal{K}_{A^\star}$*. The minimum reached is*

$$
\min_{\{\mathsf{S}:\mathcal{K}_{A^\star} \to A^\star\}} \mathsf{E}[\mathsf{I}(\mathcal{G}_{A^\star}, A, \mathcal{K}_{A^\star})] = \mathsf{E}\left[\min_{A \in A^\star} \mathsf{E}[\mathsf{I}|A, \mathcal{K}_{A^\star}]\right]. \tag{29}
$$

Proof Let us fix an arbitrary $\mathcal{K}_{A^\star} \in \mathcal{K}_{A^\star}{}^\star$. The definition of minimum implies that for all $A \in A^\star$

$$\mathsf{E}\left[\mathsf{I}|\,{}^{O}\mathsf{S}(\mathcal{K}_{A^\star}), \mathcal{K}_{A^\star}\right] \le \mathsf{E}[\mathsf{I}|A, \mathcal{K}_{A^\star}].$$

Let a decision rule S assign an action $A \in A^\star$ to the considered $\mathcal{K}_{A^\star}$. Then, the previous inequality becomes

$$\mathsf{E}[\mathsf{I}|\,{}^{O}\mathsf{S}(\mathcal{K}_{A^\star}), \mathcal{K}_{A^\star}] \le \mathsf{E}[\mathsf{I}|\mathsf{S}(\mathcal{K}_{A^\star}), \mathcal{K}_{A^\star}].$$

Let us apply unconditional expectation $\mathsf{E}[\cdot]$ acting on functions of $\mathcal{K}_{A^\star}$ to this inequality. Due to the isotonic nature of $\mathsf{E}[\cdot]$, the inequality is preserved. The the chain rule for expectation – see Proposition 8 – implies that on the right-hand side of the resulting inequality we get the unconditional expected loss corresponding to an arbitrarily chosen decision rule S. On the left-hand side the unconditional expected loss for ${}^{O}\mathsf{S}$ arises. Thus, ${}^{O}\mathsf{S} : \mathcal{K}_{A^\star} \to {}^{O}A(\mathcal{K}_{A^\star})$ is the optimal decision rule. $\square$

Proposition 10 and its proof imply no preferences if there are several globally minimising arguments ${}^{O}A(\mathcal{K}_{A^\star})$. We can use any of them or switch between them in a random manner according to the pd $\mathsf{f}(A_t|\mathcal{K}_{t-1})$, which has its support concentrated on minimisers. This is an example of a randomised causal decision rule that is modelled by a pd.

- *model of decision rule* is pd $\mathsf{f}(A|\mathcal{K}_{A^\star})$.
- *model of decision strategy* are pds $(\mathsf{f}(A_t|\mathcal{K}_{t-1}))_{t\in t^\star}$. Strategies with the same model provide the same closed loop model↰. Thus, strategy↰ can be identified with its model.
- *randomised decision rule* has its model with the support containing at least two actions.
- *randomised strategy* has at least one randomised decision rule.

Remark 3

- *We do not enter the game with ε-optimum: the existence of the various minimisers is implicitly supposed.*
- *It is worth repeating that the optimal decision rule is constructed value-wise. To get the decision rule, the minimisation should be performed for all possible instances of knowledge $\mathcal{K}_{A^\star} \in \mathcal{K}_{A^\star}{}^\star$.*
- *Often, we are interested in the optimal action for a given fixed, say observed, knowledge. Then, just a single minimisation is necessary. This is typically the case of the estimation problem. This possibility makes the main distinction from the dynamic design, when the optimal strategy, a sequence of decision rules, is searched for. In this case, see Section 11, the construction of decision rules is necessary. This makes the dynamic design substantially harder and, mostly, exactly infeasible [Fel60, Fel61].*

Estimate as Static Design

Example 5 (Point Estimate)

- *Let the behaviour be* $B = (\mathcal{G}_{A^\star}, A, \mathcal{K}_{A^\star}) = (\Theta, \hat{\Theta}, D) =$ *(unknown parameter,parameter estimate,known data). The optimal estimator↰, is searched among rules* $\mathsf{S} : \mathcal{K}_{A^\star}{}^\star \to A^\star$.
- *A performance index* $\mathsf{I}(B) = \mathsf{I}(\Theta, \hat{\Theta}, D)$, *strictly convex in* Θ, *is a "distance" of* Θ *to* $\hat{\Theta}$ *with minimum for* $\hat{\Theta} = \Theta \ \forall D \in D^\star$.
- *Proposition 10 provides the optimal estimate* ${}^{O}\hat{\Theta}(D) \in \mathrm{Arg}\min_{\hat{\Theta}\in\hat{\Theta}^\star} \mathsf{E}[\mathsf{I}(\Theta, \hat{\Theta}, D)|\hat{\Theta}, D]$.
- *Jensen inequality (27) gives* $\mathsf{E}[\mathsf{I}(\Theta, \hat{\Theta}, D)|\hat{\Theta}, D] \geq \mathsf{I}(\mathsf{E}[\Theta|\hat{\Theta}, D], \hat{\Theta}, D)$, *i.e. the minimiser is* ${}^{O}\hat{\Theta}(D) = \mathsf{E}[\Theta|\hat{\Theta}, D]$.
- *The evaluation of the optimal estimate requires specification of the posterior pd↰* $\mathsf{f}(\Theta|\hat{\Theta}, D)$. *The estimate* $\hat{\Theta}(D)$ *has no influence on the parameter* Θ: *natural conditions of DM↰* $\mathsf{f}(\Theta|D) = \mathsf{f}(\Theta|\hat{\Theta}, D)$ *are acceptable, i.e.* $\hat{\Theta}(D)$ *and* Θ *are conditionally independent↰.*

Dynamic Design

Traditional Dynamic DM Design

- We are searching for the optimal admissible strategy↰ assuming that each decision rule↰ has at least the same knowledge↰ as its predecessor. This extending knowledge models an increasing number of data available for the DM↰.
- The considered performance index↰ evaluates behaviour↰ not the strategy↰ used, i.e. the traditional DM design↰ is addressed.

Extending Knowledge: Formalisation

- The addressed dynamic design deals with the knowledge permanently extended by the by chosen actions A_t & an observation Δ_t, $t \in t^\star$,

$$\mathcal{K}_t{}^\star = (A_t{}^\star, \Delta_t{}^\star) \cup \mathcal{K}^\star_{t-1} = D_t{}^\star \cup \mathcal{K}^\star_{t-1}. \tag{30}$$

- *data record* $D_t = (A_t, \Delta_t)$ =(action,observation) enriches the knowledge $\mathcal{K}_{t-1}$ to the knowledge $\mathcal{K}_t$.
  Attaching formally the knowledge $\mathcal{K}_0$ to D^t as D^0, allows the identification

$$\mathcal{K}_{t-1} = D^{t-1}. \tag{31}$$

  Both variants are used in the text.
- The optimal admissible strategy can be found by using a stochastic version of celebrated dynamic programming [Bel67]. It is nothing but a repetitive application of Proposition 10 evolving Bellman function V($\mathcal{K}_{t-1}$) & determining actions of the constructed optimal strategy.

Proposition 11 (Dynamic Programming)

The causal strategy ${}^{O}\mathsf{S} \equiv \left({}^{O}\mathsf{S}_t : \mathcal{K}^{\star}_{t-1} \to A^{\star}_t \right)_{t \in t^\star}$ *with extending knowledge* $\mathcal{K}_t{}^{\star} = D_t{}^{\star} \cup \mathcal{K}^{\star}_{t-1}$ *minimising the expected traditional performance index* $\mathsf{E}[\mathsf{I}(B)]$ *can be constructed in a value-wise way. For every* $t \in t^\star$ *and* $\mathcal{K}_{t-1} \in \mathcal{K}_{t-1}{}^{\star}$, *it suffices to take a minimiser* ${}^{O}A(\mathcal{K}_{t-1})$ *in*

$$\mathsf{V}(\mathcal{K}_{t-1}) = \min_{A_t \in A_t{}^\star} \mathsf{E}[\mathsf{V}(\mathcal{K}_t)|A_t, \mathcal{K}_{t-1}], \; t \in t^\star, \tag{32}$$

as the t*th action* ${}^{O}A(\mathcal{K}_{t-1}) = {}^{O}\mathsf{S}_t(\mathcal{K}_{t-1})$ *of the optimal strategy* ${}^{O}\mathsf{S}$. *The functional recursion (32) is evaluated in the backward manner against the course given by the extending knowledge. The recursion starts with*

$$\mathsf{V}(\mathcal{K}_h) \equiv \mathsf{E}[\mathsf{I}(B)|\mathcal{K}_h], \tag{33}$$

$\mathcal{K}_h$ *contains knowledge available up to and including horizon* h. *The reached minimum value is*

$$\mathsf{E}[\mathsf{V}(\mathcal{K}_0)] = \min_{\mathsf{S}^\star \equiv \left\{ \left(\mathsf{S}_t : \mathcal{K}^{\star}_{t-1} \to A_t{}^\star \right)_{t \in t^\star} \right\}} \mathsf{E}[\mathsf{I}(B)].$$

Proof Let $\mathcal{K}_h$ be the knowledge available at when reaching the horizon h. The chain rule for expectation (25) and (33) imply

$$\mathsf{E}[\mathsf{I}(B)] = \mathsf{E}[\mathsf{E}[\mathsf{I}(B)|\mathcal{K}_h]] \equiv \mathsf{E}[\mathsf{V}(\mathcal{K}_h)].$$

This identity allows us to get a uniform notation. The definition of $\mathcal{K}_h$ is legitimate as A_{h+1} is not optimised. The definition of minimum and Proposition 10 imply

$$\begin{aligned}
&\min_{\left(\mathsf{S}_t:\, \mathcal{K}^\star_{t-1} \to A_t{}^\star\right)_{t\in t^\star}{}^\star} \mathsf{E}[\mathsf{V}(\mathcal{K}_h)] \\
&= \min_{\left(\mathsf{S}_t:\, \mathcal{K}^\star_{t-1} \to A_t{}^\star\right)_{t<h}{}^\star} \left(\min_{\{\mathsf{S}_h:\, \mathcal{K}_{h-1}{}^\star \to A_h{}^\star\}} \mathsf{E}[\mathsf{V}(\mathcal{K}_h)] \right) \\
&\underbrace{=}_{(29)} \min_{\left(\mathsf{S}_t:\, \mathcal{K}^\star_{t-1} \to A_t{}^\star\right)_{t<h}{}^\star} \mathsf{E}\left[\min_{A_h \in A_h{}^\star} \mathsf{E}[\mathsf{V}(\mathcal{K}_h)|A_h, \mathcal{K}_{h-1}] \right].
\end{aligned}$$

$\square$

Proof Denoting $\mathsf{V}(\mathcal{K}_{h-1}) \equiv \min_{A_h \in A_h^\star} \mathsf{E}[\mathsf{V}(\mathcal{K}_h)|A_h, \mathcal{K}_{h-1}]$, we proved the first step of the recursion and specified the start (33). The following step becomes $\min_{\{\mathsf{S}_t:\mathcal{K}_{t-1}\to A_t{}^\star\}_{t<h}{}^\star} \mathsf{E}\left[\mathsf{V}(\mathcal{K}_h)\right]$. We face the same situation with the horizon decreased by one. The procedure can be repeated until the optimal decision rule ${}^{O}\mathsf{S}_1$ is found. $\square$

- The optimisation relies on our ability to evaluate $\forall t \in t^\star$ the expectations

$$\mathsf{E}[\mathsf{V}(\mathcal{K}_t)|A_t, \mathcal{K}_{t-1}] = \int_{\Delta_t{}^\star} \mathsf{V}(\Delta_t, A_t, \mathcal{K}_{t-1}) \mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1})\, \mathrm{d}\Delta_t.$$

- The freedom in the choice of the performance index implies that the set of possible functions $\mathsf{V}(B) = \mathsf{V}(\Delta_t, A_t, \mathcal{K}_{t-1})$ is extremely rich and thus the full knowledge of pds $\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1})$ is generally needed.

Predictor of Observations

- The collection of pds $\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1})$ relates the observation↰ Δ_t to the action↰ A_t and its knowledge↰ $\mathcal{K}_{t-1}$. Each pd↰ predicts observable response Δ_t of the system↰ to A_t and $\mathcal{K}_{t-1}$. This leads to the notion predictor.
- *predictor* of observations is the collection of pds needed for the optimal design↰,
$$\left(\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1})\right)_{t \in t^\star}. \tag{34}$$
Often, the sole term predictor is used. The context clarifies the meaning. Sometime, the alternative term is used:
- *predictive pd* stresses that the predictor↰ is whole pd not only its characteristics (like expectation or variance).

Data-Driven Design

Generally, the behaviour↰ contains hidden quantities.

- *hidden quantity* is a part of behaviour↰ B. B consists of potentially observable Δ^h (observations) and optional actions A^h. They form observable data records D^h. Behaviour B may contain hidden quantities X^h that are never observed directly. While observation and action realisations move data record↰ from ignorance↰ to knowledge↰, the hidden quantities stay within ignorance permanently $X^h \in \mathcal{G}_\tau, \ \tau \in t^\star$.
- Hidden quantities influence the optimal design "only" through the terminal condition (33) of dynamic programming, see Proposition 11. Its evaluation uses the conditional pd↰ $\mathsf{f}(X^h|\mathcal{K}_h)$, see Section 7.
- Having $\mathsf{V}(\mathcal{K}_h) = \mathsf{V}(D^h)$, predictor↰ of observations is the only model needed in the design. For $t < h$, we face
- *data-driven design* whose performance index↰ depends on data

$$\mathsf{I}(B) \equiv \mathsf{I}(\Delta^h, A^h) = \mathsf{I}(D^h) \equiv \mathsf{I}(\mathcal{K}_h). \tag{35}$$

Proposition 12 (Data-Driven Design: Additive Performance Index)

In the data-driven design the optimal admissible strategy ${}^{O}\mathsf{S} \equiv \left({}^{O}\mathsf{S}_t : \mathcal{K}^\star_{t-1} \to A_t{}^\star \right)_{t \in t^\star}$ *acting on an extending knowledge* $\mathcal{K}_t{}^\star = D^\star_t \cup \mathcal{K}^\star_{t-1}$ *is searched for.* ${}^{O}\mathsf{S}$ *is to minimise , cf. (31),*

- *additive performance index*

$$\mathsf{E}\left[\mathsf{I}\left(D^h\right)\right] \equiv \mathsf{E}\left[\sum_{t\in t^\star} \mathsf{z}(\Delta^t, A^t)\right] \equiv \mathsf{E}\left[\sum_{t\in t^\star} \mathsf{z}(D^t)\right] \equiv \mathsf{E}\left[\sum_{t\in t^\star} \mathsf{z}(\mathcal{K}_t)\right] \qquad (36)$$

- *partial performance index is* $\mathsf{z}(\Delta^t, A^t) = \mathsf{z}(D^t) = \mathsf{z}(\mathcal{K}_t) \geq 0.$

The optimal strategy ${}^{O}\mathsf{S}$ *can be constructed in the value-wise way. For all* $\mathcal{K}_{t-1} \in \mathcal{K}_{t-1}{}^\star$, $t \in t^\star$, *a minimising argument* ${}^{O}A(\mathcal{K}_{t-1})$ *in*

$$\mathsf{V}(\mathcal{K}_{t-1}) = \min_{A_t \in A_t{}^\star} \mathsf{E}[\mathsf{z}(\mathcal{K}_t) + \mathsf{V}(\mathcal{K}_t) | A_t, \mathcal{K}_{t-1}], \ t \in t^\star, \qquad (37)$$

is the optimal action, ${}^{O}A(\mathcal{K}_{t-1}) = {}^{O}\mathsf{S}_t(\mathcal{K}_{t-1})$*. The recursion (37) runs against the course of knowledge extension, starting from* $\mathsf{V}(\mathcal{K}_h) = 0$ *and reaching the minimum* $\mathsf{E}[\mathsf{V}(\mathcal{K}_0)]$.

Proof It follows exactly the line of Proposition 11 with a modified definition of the function $\mathsf{V}(\cdot)$

$$\mathsf{V}(\mathcal{K}_{t-1}) \equiv \min_{\left(\mathsf{S}_\tau:\mathcal{K}^\star_{\tau-1}\to A_\tau{}^\star\right)_{\tau\geq t}{}^\star} \sum_{\tau\geq t} \mathsf{E}\left[\mathsf{z}(\mathcal{K}_\tau)\middle|\,\mathcal{K}_{t-1}\right]. \qquad (38)$$

$\square$

- *value function* is an accepted name for the function $\mathsf{V}(\cdot)$ evolving in the general dynamic programming, Proposition 11.
- *Bellman function* is an alternative name of the value function.
- *loss-to-go* is another wide-spread name of the value function in the special case (38).
- Non-negativity of the partial preference index can be replaced by boundedness from below.

Asymptotic of the Design

- The asymptotic of the dynamic programming is inspected for the horizon $h \to \infty$ within this section.
- The outlined analysis serves us only as a motivation for approximate design, see Section 29. Thus, technicalities are suppressed as much as possible.
- The data-driven design with an additive performance index (36) is considered only.
- The general, data-dependent performance index can always be converted into the additive form by defining the partial performance index

$$z(\mathcal{K}_t) = z(D^t) = z(\Delta^t, A^t) = \begin{cases} \mathsf{I}(\Delta^h, A^h) & \text{if } t = h, \\ 0 & \text{otherwise} \end{cases}. \qquad (39)$$

Time-Invariant Data-Driven Design

We deal with a simpler but still useful data-driven design↰ assuming existence of

- *information state* , which is an observed finite-dimensional array replacing in a sufficient way the knowledge↰ $\mathcal{K}_{t-1}$, i.e. $\mathsf{E}[\bullet|A_t, \mathcal{K}_{t-1}] = \mathsf{E}[\bullet|A_t, X_{t-1}]$.
- We assume that the partial performance index↰ depends on the information state X_t and the action↰ A_t only, i.e. $\mathsf{z}(\Delta^t, A^t) \equiv \mathsf{z}(X_t, A_t)$ and the considered performance index↰ is

$$\mathsf{I}(\mathcal{K}_h) = \mathsf{I}(D^h) = \mathsf{I}(\Delta^h, A^h) = \sum_{t \in t^\star} \mathsf{z}(X_t, A_t). \tag{40}$$

- Asymptotic analysis makes sense only when a meaningful solution of the DM design↰ exists even for an unbounded decision horizon↰.

Stabilising Strategy

- *stabilising strategy* is defined as follows: let us consider sequence of DM designs with the growing horizon $h \to \infty$, i.e. with extending sets ${}^{h}t^{\star} \equiv \{1, \ldots, h\}$ of time indices. The infinite sequence of decision rules

$$\{\mathsf{S}_t : \mathcal{K}_{t-1} \to A_t^{\star}\}_{t \in {}^{\infty}t^{\star} \equiv \{1,2,\ldots,\}}$$

is called the stabilising strategy if there is a finite constant c such that the expectation of the (non-negative) partial performance index

$$\mathsf{E}[\mathsf{z}(X_t, A_t)|A_t, \mathcal{K}_{t-1}] \le c < \infty, \ t \in {}^{\infty}t^{\star} \equiv \{1, 2, 3, \ldots\}. \tag{41}$$

- Obviously, the expected performance index with the growing decision horizon grows to infinity as (in generic case) it is a sum of positive terms.
- Consequently, a change of finite number of decision rules forming the strategy has no influence on the expected performance index.

Asymptotically Optimal Strategy is Stationary

- For $h \to \infty$, the influence of DM-rules' changes on the expected performance index↰ diminishes and the optimal strategy↰ is stationary.
- *stationary strategy* means a DM strategy↰ formed by a repetitive use of the same rule. Its (approximate) evaluation is simpler than that of a strategy↰ with time-varying rules.

Proposition 13 (Asymptotic Design)

Let a stabilising strategy↰ exist with the expected partial performance index↰ (depending on action A_t and a finite-dimensional information state↰ X_t) bounded by a $c < \infty$. Then, for $h \to \infty$, the optimal strategy↰ can be chosen as stationary strategy↰. Actions generated by the decision rule↰ defining it are minimising arguments in the formal analogy of (37)

$$ {}^{\infty}\mathsf{V}(X_{t-1}) + {}^{\infty}C = \min_{A_t \in A_t^{\star}} \mathsf{E}\left[\mathsf{z}(X_t, A_t) + {}^{\infty}\mathsf{V}(X_t) \middle| A_t, X_{t-1}\right] \tag{42} $$

with a constant ${}^{\infty}C \le c$ and a time-invariant Bellman function↰ ${}^{\infty}\mathsf{V}(X)$.

Proof

- Let us take any finite horizon↰ h and, within this horizon↰, denote ${}^{h}\tilde{V}(\mathcal{K}_{t-1}) \equiv {}^{h}\tilde{V}(X_{t-1})$ the optimal loss-to-go↰.
- Let us define ${}^{h}C$ as the smallest value such that

$$ {}^{h}V(X_t) \equiv {}^{h}\tilde{V}(X_t) - (h-t)\, {}^{h}C $$

  is bounded from above for $h \to \infty$ and a fixed $t \in {}^{\infty}t^{\star}$, $X_t \in X^{\star}$.
- Obviously, the optimal strategy↰ cannot lead to a higher expected performance index↰ than any stabilising strategy↰. Thus, the optimal strategy has to also be a stabilising strategy↰. Thus, ${}^{h}C \leq c$ and $\overline{\lim}_{h\to\infty} {}^{h}C = {}^{\infty}C$ exists.
- The optimisation is uninfluenced if we subtract the value ${}^{h}C$ from each partial performance index↰. For arbitrary fixed t, X_t, the corresponding modified loss-to-go↰ ${}^{h}V(X_t)$, is bounded from above. ${}^{h}V(X_t) = {}^{h}\tilde{V}(X_t) - (h-t)\, {}^{h}C$ is the difference between a pair of monotonous sequences (indexed by h). Thus, a finite limit ${}^{\infty}V(X_t) = \lim_{h\to\infty} {}^{h}V(X_t)$ exists.

□

Proof

- The modified loss-to-go↰ fulfills the equation

$$^{h}\mathsf{V}(X_{t-1}) + {}^{h}C = \min_{A_t \in A_t^{\star}} \mathsf{E}\left[\mathsf{z}(X_t, A_t) + {}^{h}\mathsf{V}(X_t) | A_t, X_{t-1}\right].$$

- Existence and finiteness of the involved limits imply that the asymptotic version of the Bellman equation is fulfilled, too,

$$^{\infty}\mathsf{V}(X_{t-1}) + \overline{\lim}_{h\to\infty} {}^{h}C = \min_{A_t \in A_t^{\star}} \mathsf{E}\left[\mathsf{z}(X_t, A_t) + {}^{\infty}\mathsf{V}(X_t) | A_t, X_{t-1}\right].$$

  Limits of ${}^{h}\mathsf{V}(X_t)$ exist and, thus, $\overline{\lim}_{h\to\infty} {}^{h}C = \lim_{h\to\infty} {}^{h}C = {}^{\infty}C$.
- The identical optimisation is performed for each $t < \infty$. Thus, it provides the same decision rule↰ for each t: the optimal strategy is a stationary one.

□

A Few Comments

Remark 4

- *The value function↰ is unique up to a shift.*
- *Solutions of the Bellman equation for a growing horizon↰ h represent successive approximations for solving its stationary version (42).*
- *iterations in strategy space , [Kus71], provide an alternative way of finding the solution. Essentially, a stabilising stationary strategy* $\mathsf{S} \in \mathsf{S}^\star$ *is selected and the linear equation*

$$\mathsf{V}(X) + C = \mathsf{E}[\mathsf{z}(\tilde{X}, \mathsf{S}(X)) + \mathsf{V}(\tilde{X}) | \mathsf{S}(X), X]$$

  is solved for the function $\mathsf{V}(\cdot)$ *and constant* C*. Then, a new approximating strategy is found in the value-wise way* $\mathsf{S}(X) \in \mathrm{Arg}\min_{A \in A^\star} \mathsf{E}[\mathsf{z}(\tilde{X}, A) + \mathsf{V}(\tilde{X}) | A, X]$ *with such a* $\mathsf{V}(\cdot)$.
- *Under general conditions, the newly found strategy is stabilising and iterations may be repeated until the guaranteed convergence.*

Stationary DM with Markov Predictor

Example 6 (DM with Markov Predictor)

- *The time-invariant data-driven design↰ with observation↰ $\Delta^\star$ and action↰ $A^\star$ spaces having finite cardinalities is considered. A time-invariant partial performance index↰* $\mathsf{z}(\Delta_t, A_t)$, $\Delta_t \in \Delta^\star$, $A_t \in A^\star$ *(a finite table) determines the selected additive performance index.*
- *The past observation↰* Δ_{t-1} *is assumed to be information state↰, i.e.* $\mathsf{E}[\bullet|A_t, \mathcal{K}_{t-1}] = \mathsf{E}[\bullet|A_t, \Delta_{t-1}]$. *It means that the system is modelled by (controlled) Markov chain [Kus71].*
- *Propositions 12 and 13 directly imply that the loss-to-go↰ is a time-invariant finite table* $\mathsf{V}(\Delta_t)$, $\Delta_t \in \Delta^\star$. *The optimal decision rule↰* $\mathsf{f}(A_t|\mathcal{K}_{t-1}) = \mathsf{f}(A_t|\Delta_{t-1})$, *determining the stationary strategy↰, is concentrated on minimising argument* ${}^{O}A_t = {}^{O}A(\Delta_{t-1})$ *in*

$$\mathsf{V}(\Delta_{t-1}) + C = \min_{A \in A^\star} \mathsf{E}[\mathsf{z}(\Delta_t, A_t) + \mathsf{V}(\Delta_t)|A_t, \Delta_{t-1}], \ \forall \Delta_{t-1} \in \Delta^\star.$$

Learning

Why Learning Is Needed?

- Behaviour $B \in B^{\star}$ includes generally an hidden quantity↰ X^h, which is never observed directly but influences observation↰.
  Question arises how to get the predictor↰ (34) needed in the optimal DM↰, see Proposition 11.
- Generally, the performance index↰ depends on X^h. For instance, it happens when we want to estimate an unknown quantity.
  Generally, DM↰ wants to influence hidden quantity↰ in spite of the fact that we do not observe them directly,
- In both cases, the general dynamic programming, Proposition 11 needs the pd $\mathsf{f}(X^h|\mathcal{K}_h)$, an estimate of hidden quantity↰, for evaluation of the initial condition (33) of dynamic programming.
- Here we describe how to get both the predictor↰ and the estimate of hidden quantities. The solved problem, known as nonlinear filtering↰ [Jaz70], is of an independent interest as its solution provides a consistent formal prescriptive model of learning.

The joint pd↰ $f(B)$ describing observed, opted and hidden quantities is constructed from the following elements.

- *observation model* relates observations Δ_t to hidden X^t, to an action↰ A_t and its knowledge↰ $\mathcal{K}_{t-1}$

$$\left(f(\Delta_t|X_t, A_t, \mathcal{K}_{t-1}) \equiv f(\Delta_t|X^t, A_t, \mathcal{K}_{t-1})\right)_{t \in t^\star}. \quad (43)$$

  Unlike the predictor↰, the observation model contains an unknown hidden quantity↰ $X_t \in X_t^\star \subset \mathcal{G}_\tau, \ \forall \tau \in t^\star$. kalman

- *time evolution model* relates the hidden quantities $X^h \in X^{h^\star}$

$$\left(f(X_t|X_{t-1}, A_t, \mathcal{K}_{t-1}) \equiv f(X_t|X^{t-1}, A_t, \mathcal{K}_{t-1})\right)_{t \in t^\star}. \quad (44)$$

Remark 5

The conditional independence, required by (43) for observations and by (44) for time evolution models is unrestrictive as it can always be met by a suitable re-definition of hidden ($X_t = X^t$).

Natural Conditions of DM and Prior Pd

The processing is made under natural conditions of DM.

- *natural conditions of DM* formally express [Pet81] that quantities X^h are unknown to the strategies considered. They postulate independence of A_t and X^{t-1} when conditioned on $\mathcal{K}_{t-1}$

$$f(A_t|X^{t-1}, \mathcal{K}_{t-1}) = f(A_t|\mathcal{K}_{t-1}) \quad \underbrace{\Leftrightarrow}_{\text{Proposition 5}} \quad f(X^{t-1}|A_t, \mathcal{K}_{t-1}) = f(X^{t-1}|\mathcal{K}_{t-1}). \tag{45}$$

The inspected filtering starts from

- *prior pd* $f(X_0)$ that expresses the prior knowledge about the initial hidden quantity X_0. Thus, it fulfills

$$f(X_0) \equiv f(X_0|\mathcal{K}_0) \underbrace{=}_{(45)} f(X_0|A_1, \mathcal{K}_0). \tag{46}$$

Are Natural Conditions of DM Natural?

Remark 6

- *Often, the unknown quantities X_t together with the action↰ A_t are assumed to describe the involved conditional pds fully. Then, $\mathcal{K}_{t-1}$ is omitted.*
- *The natural conditions of DM↰ express the assumption that $X_t \notin \mathcal{K}_{\tau-1}\ \forall\tau, \forall t \in t^\star$. Thus, values of X^{t-1} cannot be used by the decision rules forming the admissible strategy↰. Alternatively, we cannot gain information about X^{t-1} from the action↰ A_t if the corresponding observation↰ Δ_t (the corresponding reaction of the system↰) are unavailable.*
- *The hidden X_τ $\tau \geq t$ can be influenced by A_t.*
- *The natural conditions of DM↰ are "naturally" fulfilled by strategies we are designing. They have to be checked when the data record↰s influenced by an "externally chosen" strategy↰ are processed.*

Proposition 14 (Generalised Bayesian Filtering)

Under natural conditions of DM↰, the predictor↰ (34) reads

$$f(\Delta_t|A_t,\mathcal{K}_{t-1}) = \int_{X_t^\star} f(\Delta_t|X_t,A_t,\mathcal{K}_{t-1}) f(X_t|A_t,\mathcal{K}_{t-1}) dX_t. \quad (47)$$

It needs generalised Bayesian

- *filtering , which labels the evolution of the pd $f(X_t|A_t,\mathcal{K}_{t-1})$ from the prior pd↰ $f(X_0)$. The filtering consists of the pairs* kalman
- *data updating extends the knowledge $\mathcal{K}_{t-1}$ by the data record↰= D_t=(action↰,observation↰)=(A_t,Δ_t)* fkalman

$$\begin{aligned} f(X_t|\mathcal{K}_t) &= \frac{f(\Delta_t|X_t,A_t,\mathcal{K}_{t-1}) f(X_t|A_t,\mathcal{K}_{t-1})}{f(\Delta_t|A_t,\mathcal{K}_{t-1})} \quad (48) \\ &\propto f(\Delta_t|X_t,A_t,\mathcal{K}_{t-1}) f(X_t|A_t,\mathcal{K}_{t-1}). \end{aligned}$$

- *time updating reflects evolution $X_t \to X_{t+1}$, A_{t+1} given,*

$$f(X_{t+1}|A_{t+1},\mathcal{K}_t) = \int_{X_t^\star} f(X_{t+1}|X_t,A_{t+1},\mathcal{K}_t) f(X_t|\mathcal{K}_t)\, \mathrm{d}X_t. \quad (49)$$

Proof Sequential use of the marginalisation↰, chain rule↰ and Proposition 5 imply

$$\begin{aligned} \mathsf{f}(\Delta_t|A_t,\mathcal{K}_{t-1}) &= \int_{X_t^\star} \mathsf{f}(\Delta_t, X_t|A_t,\mathcal{K}_{t-1})\,\mathrm{d}X_t \\ &= \int_{X_t^\star} \mathsf{f}(\Delta_t|X_t, A_t,\mathcal{K}_{t-1})\mathsf{f}(X_t|A_t,\mathcal{K}_{t-1})\,\mathrm{d}X_t. \end{aligned}$$

The data updating↰ coincides with the Bayes rule↰. The time updating↰ results from the marginalisation↰, the chain rule↰, and the natural conditions of DM↰ implying $\mathsf{f}(X_t|A_{t+1},\mathcal{K}_t) = \mathsf{f}(X_t|\mathcal{K}_t)$. □

Remark 7

The described filtering↰ is called generalised to distinguish a nonstandard use of the terms Bayesian filtering and predictions [Jaz70]. Without this adjective, they are understood as specific DM↰ problems. The "generalisation" means that the conditional pds needed for these tasks are evaluated only. They serve for solving a whole class of DM↰ problems.

Filtering of Noisy Markov Chain

Example 7 (Filtering of Markov Chain)

- *Let the set of hidden* $\Theta^\star$ *and observations* $\Delta^\star$ *have finite cardinalities and no actions are present* $A^\star = \emptyset$.
- *Let us have a given prior pd* $\mathsf{f}(X_0|\mathcal{K}_0)$, *a time and data independent time evolution model*↰ $\mathsf{f}(X_t|X_{t-1})$ *and an observation model*↰ multinomfilter $\mathsf{f}(\Delta_t|X_t)$. *All are tables, pds with respect to a counting measure.*
- *Proposition 14 implies that the predictive pd and posterior pd*↰ *evolve for* $t \in t^\star$ *and* $\mathcal{K}_{t-1} = \Delta^{t-1}$ *as follow*

$$\mathsf{f}(\Delta_t|\mathcal{K}_{t-1}) = \sum_{X_t \in X^\star} \mathsf{f}(\Delta_t|X_t)\mathsf{f}(X_t|\mathcal{K}_{t-1}) \text{ (predictive pd↰)}$$

$$\mathsf{f}(X_t|\mathcal{K}_t) = \frac{\mathsf{f}(\Delta_t|X_t)\mathsf{f}(X_t|\mathcal{K}_{t-1})}{\sum_{X_t \in X^\star} \mathsf{f}(\Delta_t|X_t)\mathsf{f}(X_t|\mathcal{K}_{t-1})} \text{ (data updating↰)}$$

$$\mathsf{f}(X_{t+1}|\mathcal{K}_t) = \sum_{X_t \in X^\star} \mathsf{f}(X_{t+1}|X_t)\mathsf{f}(X_t|\mathcal{K}_t) \text{ (time updating↰)}.$$

More Comments

Remark 8

- *Filtering extrapolates knowledge↰ into ignorance↰ assuming that the rule generating X_t does not change: a knowledge↰ can accumulate only with fixed rules governing behaviour.*
- *Under natural conditions of DM↰, the closed loop model↰ factorises*

$$\mathsf{f}_{\mathsf{S}}(B) = \underbrace{\overbrace{\mathsf{f}(X_0)}^{\text{prior pd}} \prod_{t\in t^\star} \overbrace{\mathsf{f}(\Delta_t, X_t|X_{t-1}, A_t, \mathcal{K}_{t-1})}^{\text{observation}\times\text{time evolution pds}}}_{\text{system model M}} \underbrace{\prod_{t\in t^\star} \mathsf{f}(A_t|\mathcal{K}_{t-1})}_{\text{strategy S}} = \mathsf{M}(B)\mathsf{S}(B), \tag{50}$$

reflecting that the compared strategies↰ work with a common

- *system model* $\mathsf{M} = \mathsf{f}(X_0)\prod_{t\in t^\star}\mathsf{f}(\Delta_t, X_t|X_{t-1}, A_t, \mathcal{K}_{t-1})$ *and identification of the model of strategy with strategy↰ are used.*
- *The presented accumulation of knowledge and its extrapolation represent a good prescriptive model of learning.*

Filtering in Service of Dynamic Programming

- The filtering is often of independent interest but the construction of the predictive pd and of the pd needed in (33) are our key motivation for its formulation and solution.
- Under the adopted conditions, the pd (33), needed for initiation of the dynamic programming, evaluates recursively

$$
f(X^h|\mathcal{K}_h) \underbrace{=}_{(22)} \frac{f(X^h, D_h|\mathcal{K}_{h-1})}{f(D_h|\mathcal{K}_{h-1})} = f(X^{h-1}|\mathcal{K}_{h-1}) \tag{51}
$$

$$
\times \frac{f(\Delta_h|X^h, A_h, \mathcal{K}_{h-1}) f(X_h|X^{h-1}, A_h, \mathcal{K}_{h-1}) f(A_h|X^{h-1}, \mathcal{K}_{h-1})}{f(D_h|\mathcal{K}_{h-1})}
$$

$$
\underbrace{=}_{(43),(44),(45)} f(X^{h-1}|\mathcal{K}_h) \times \frac{f(\Delta_h|X_t, A_h, \mathcal{K}_{h-1}) f(X_h|X_{h-1}, A_h, \mathcal{K}_{h-1})}{f(\Delta_h|A_h, \mathcal{K}_{h-1})}.
$$

- The derived recursion uses the observation model and the time evolution model. It can be formally repeated until arriving at the prior pd as starting point $f(X_0|\mathcal{K}_0) \equiv f(X_0)$.

Remark 9

- *Under the natural conditions of DM↰, filtering↰ relies on the knowledge of actions and not of the knowledge of the strategy↰* S *generating them. It is important when we learn while the decision loop is closed, especially, by a human decision maker.*
- *The time evolution model↰* $\mathsf{f}(X_t|X_{t-1}, A_t, \mathcal{K}_{t-1})$ *as well as the observation model↰* $\mathsf{f}(\Delta_t|X_t, A_t, \mathcal{K}_{t-1})$ *have to result from a theoretical system modelling. Modelling uses both field knowledge, like conservation laws, , e.g., [KSVZ88], and approximation capabilities [Hay94] of a model family. The prior pd↰* $\mathsf{f}(X_0)$ *quantifies either expert knowledge or situational analogy, see Section 22.*
- *The observations, the only bridge to reality, enter the evaluations in the data-updating step only when the newest (action↰,observation↰) pair is processed. This simple fact is important for approximation of the time evolution model↰, see Section 28.*

Summarising Comments on Filtering

- The described Bayesian filtering↰ combines the prior knowledge↰ quantified by the prior pd↰ $f(X_0)$, the theoretical knowledge of the specific fields described by the observation model↰ $f(\Delta_t|X_t, A_t, \mathcal{K}_{t-1})$, the time evolution model↰ $f(X_t|X_{t-1}, A_t, \mathcal{K}_{t-1})$ and the data records $D^h = (A^h, \Delta^h)$ by using coherent deductive calculus with pds.
- This combination of information sources is a powerful, internally consistent, framework describing the essence of learning. Due to its deductive structure, an important assurance is gained:

  The incorrect modelling or non-informative data can only be blamed for a possible failure of the learning process.

  Thus, the errors caused by an improper choice of the learning method are avoided.

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Bayesian Estimation

This section deals with a special version of filtering called estimation.

- *estimation* is filtering, which arises when the hidden quantities X_t are time invariant
$$X_t = \Theta, \ \forall t \in t^\star. \tag{52}$$
- *unknown parameter* is the common value of time-invariant hidden quantities. The time evolution model of the unknown parameter is $\mathsf{f}(X_t|X_{t-1}, A_t, \mathcal{K}_{t-1}) = \delta(X_t - X_{t-1})$.
- *Dirac delta* $\delta(\cdot)$ is a formal pd of the measure fully concentrated on zero argument. For a formally correct handling consult [Vla79].
- A direct specialisation of Proposition 14 provides the solution of the Bayesian estimation.

Proposition 15 (Bayesian Estimation)

Let natural conditions of DM↰ be met and hidden $X_t = \Theta \in \Theta^\star \subset \mathcal{G}_\tau$, $\forall t, \tau \in t^\star$ *be time invariant. Then, the predictive pd↰ reads*

$$\mathsf{f}(\Delta_t|A_t,\mathcal{K}_{t-1}) = \int_{\Theta^\star} \mathsf{f}(\Delta_t|\Theta,A_t,\mathcal{K}_{t-1})\mathsf{f}(\Theta|\mathcal{K}_{t-1})\,\mathrm{d}\Theta. \quad (53)$$

It uses the generalised Bayesian

- *parameter estimation , which evolves*
- *posterior pd* $\mathsf{f}(\Theta|\mathcal{K}_{t-1})$ *.*
- *parameter estimate of the unknown parameter↰* Θ *is the posterior pd.*

Its evolution – uninfluenced by A_t *– coincides with the data updating↰ (48)*

$$\mathsf{f}(\Theta|\mathcal{K}_t) = \frac{\mathsf{f}(\Delta_t|\Theta,A_t,\mathcal{K}_{t-1})\mathsf{f}(\Theta|\mathcal{K}_{t-1})}{\mathsf{f}(\Delta_t|A_t,\mathcal{K}_{t-1})} \propto \mathsf{f}(\Delta_t|\Theta,A_t,\mathcal{K}_{t-1})\mathsf{f}(\Theta|\mathcal{K}_{t-1}) \quad (54)$$

initiated by the prior pd↰ $\mathsf{f}(\Theta) \equiv \mathsf{f}(\Theta|A_1,\mathcal{K}_0) = \mathsf{f}(\Theta|\mathcal{K}_0)$.

Batch (Non-Recursive) Parameter Estimation

Proposition 16 (Batch Parameter Estimation)

Under natural conditions of DM↰, the (generalised) parameter estimate↰ allows the batch evaluation of the posterior pd

$$\mathsf{f}(\Theta|\mathcal{K}_t) = \frac{\prod_{\tau\leq t} \mathsf{f}(\Delta_\tau|\Theta, A_\tau, \mathcal{K}_{\tau-1})\mathsf{f}(\Theta)}{\int_{\Theta^\star} \prod_{\tau\leq t} \mathsf{f}(\Delta_\tau|\Theta, A_\tau, \mathcal{K}_{\tau-1})\mathsf{f}(\Theta)\, \mathrm{d}\Theta} \equiv \frac{\mathsf{L}(\Theta, \mathcal{K}_t)\mathsf{f}(\Theta)}{\mathsf{J}(\mathcal{K}_t)}. \tag{55}$$

- *likelihood* $\mathsf{L} : \Theta^\star \to [0, \infty]$ *is defined*

$$\mathsf{L}(\Theta, \mathcal{K}_t) \equiv \prod_{\tau\leq t} \mathsf{f}(\Delta_\tau|A_\tau, \mathcal{K}_{\tau-1}, \Theta) \text{ for a fixed knowledge↰} \tag{56}$$

- *Recursive evaluation of the likelihood↰ coincides with that for the non-normalised posterior pd↰ (54) but starts from* $\mathsf{L}(\Theta, \mathcal{K}_0) \equiv 1$.
- *The normalisation factor* $\mathsf{J}(\cdot)$ *is*

$$\mathsf{J}(\mathcal{K}_t) = \int_{\Theta^\star} \mathsf{L}(\Theta, \mathcal{K}_t)\mathsf{f}(\Theta)\, \mathrm{d}\Theta \Rightarrow \mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1}) = \frac{\mathsf{J}(\mathcal{K}_t)}{\mathsf{J}(\mathcal{K}_{t-1})}. \tag{57}$$

Proof It exploits the calculus with pds: marginalisation↰, chain rule↰, and Bayes rule↰, Proposition 5, under the natural conditions of DM↰ (45). □

Remark 10

- *parametric model is the alternative name of the observation model*↰ $\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1})$ *used whenever the estimation problem is considered, i.e. when the hidden quantities* X_t *are time invariant.*
- *The recursive evolution of the pd* $\mathsf{f}(\Theta|\mathcal{K}_{t-1})$ *allows us to interpret the posterior pd*↰ *as the prior pd*↰ *before processing new data records.*
- *The data inserted into the parametric model*↰ *corrects the subjectively chosen prior pd*↰ $\mathsf{f}(\Theta)$. *The posterior pd*↰ $\mathsf{f}(\Theta|\mathcal{K}_{t-1})$ *reflects both objective and subjective knowledge pieces. If the data are informative, the relative contribution of the single subjective factor* $\mathsf{f}(\Theta)$ *to the posterior pd decreases with increasing* t *as the likelihood*↰ $\mathsf{L}(\Theta, \mathcal{K}_t)$ *contains* t *"objective" factors (56).*

Further Remarks On Parameter Estimation

Remark 11

- *Zero values are preserved by multiplication. Thus, the posterior pd↰ re-distributes the probability mass only within the support↰ of the prior pd↰, i.e. within the set* $\mathrm{supp}\,[\,\mathsf{f}(\Theta)] \equiv \{\Theta \in \Theta^\star : \mathsf{f}(\Theta) > 0\}$. *Consequently, the prior pd may serve for specification of hard bounds on possible parameter values, but it does not allow us to "learn" out of the* $\mathrm{supp}\,[\,\mathsf{f}(\Theta)]$.
- *Unknown parameter is always in the estimator ignorance↰.*
- *Under the natural conditions of DM↰ (45), the action values are used in estimation, not the strategy↰ generating them.*
- *The parameter* Θ *is usually finite-dimensional. Exceptionally, we deal with potentially infinite-dimensional parameter. It means that the number of unknown quantities is finite but increases without limits. This case is often called nonparametric estimation.*

Asymptotic of Estimation

- The analysis outlined here serves us primarily for interpretation of estimation results when none of the considered parametric model↰s is the objective pd↰, which is refereed here as ${}^{o}\mathsf{f}(B)$. The result can be directly used for constructions of approximate learning and design.
- Specifically, the predictor↰ of observations corresponding to the objective pd↰ ${}^{o}\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1})$ is related to the predictive pd↰ $\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1})$ – obtained through the parameter estimation↰; see Proposition 15.
- Similarly to the asymptotic design, Section 12, all technicalities are suppressed as much as possible.

The notion of the Kullback-Leibler divergence, KLD↰, [KL51] measuring proximity of a pair of pds serves us for the asymptotic analysis as well as for fully probabilistic design discussed in Section 16.

- *KLD* Kullback–Leibler divergence $\mathsf{D}(\mathsf{f}||\tilde{\mathsf{f}})$, abbreviated KLD, compares a pair of pds $\mathsf{f}, \tilde{\mathsf{f}}$ acting on a common domain $X^\star$. It is defined by the formula

$$\mathsf{D}(\mathsf{f}||\tilde{\mathsf{f}}) \equiv \int_{X^\star} \mathsf{f}(X) \ln\left(\frac{\mathsf{f}(X)}{\tilde{\mathsf{f}}(X)}\right) \mathrm{d}X. \tag{58}$$

Its asymmetry is stressed by referring to it as the KLD of f on $\tilde{\mathsf{f}}$.

Basic Properties of KLD

Proposition 17 (Properties of KLD)

Let $\mathsf{f}, \tilde{\mathsf{f}}$ *be pds (Radon–Nikodým derivatives) acting on a set* $X^\star$*. It holds*

- $\mathsf{D}(\mathsf{f}||\tilde{\mathsf{f}}) \geq 0$,
- $\mathsf{D}(\mathsf{f}||\tilde{\mathsf{f}}) = 0$ *iff* $\mathsf{f} = \tilde{\mathsf{f}}$ $\mathrm{d}X$*-almost surely*
- $\mathsf{D}(\mathsf{f}||\tilde{\mathsf{f}}) = \infty$ *iff on a set of a positive dominating measure* $\mathrm{d}X$*, it holds* $\mathsf{f} > 0$ *and* $\tilde{\mathsf{f}} = 0$,
- $\mathsf{D}(\mathsf{f}||\tilde{\mathsf{f}}) \neq \mathsf{D}(\tilde{\mathsf{f}}||\mathsf{f})$ *and the KLD does not obey triangle inequality,*
- $\mathsf{D}(\mathsf{f}||\tilde{\mathsf{f}})$ *is invariant with respect to a sufficient mapping* $\Upsilon : X^\star \to Y^\star$*. Note that any bijective mapping* Υ *is sufficient.*

Proof See, for instance, [Vaj82]. □

- The asymptotic analysis exploits the notion of entropy rate↰, a slight generalisation of KLD↰, [CK86]. It measures the divergence of the objective predictor↰ ${}^{o}\mathsf{f}(\Delta_\tau|A_\tau, \mathcal{K}_{\tau-1})$ on a parametric model↰ $\mathsf{f}(\Delta_\tau|\Theta, A_\tau, \mathcal{K}_{\tau-1})$. For a behaviour↰ realisation↰ $B = (\Theta, \mathcal{K}_\infty)$.
- *entropy rate* is, for each $\Theta \in \Theta^\star$, defined

$$\mathsf{D}_\infty\left({}^{o}\mathsf{f}||\,\Theta\right) \equiv \overline{\lim}_{t\to\infty} \mathsf{D}_t\left({}^{o}\mathsf{f}||\,\Theta\right) \tag{59}$$

$$\equiv \overline{\lim}_{t\to\infty} \frac{1}{t} \sum_{\tau \le t} \int_{\Delta_\tau^\star} {}^{o}\mathsf{f}(\Delta_\tau|A_\tau, \mathcal{K}_{\tau-1}) \ln\left(\frac{{}^{o}\mathsf{f}(\Delta_\tau|A_\tau, \mathcal{K}_{\tau-1})}{\mathsf{f}(\Delta_\tau|\Theta, A_\tau, \mathcal{K}_{\tau-1})}\right) \mathrm{d}\Delta_\tau.$$

- Non-negativity of KLD↰, Proposition 17, implies that the definition is correct and that $\mathsf{D}_\infty\left({}^{o}\mathsf{f}||\,\Theta\right) \in [0, \infty]$.

Asymptotic of Estimation

Proposition 18 (Basic Result on the Estimation Asymptotic)

Let the natural conditions of decision making (45) hold. For almost all $\Theta \in \Theta^\star$*, let there exist positive* $\underline{C}_\Theta, \overline{C}_\Theta$ *uniformly bounded by a finite* c*, i.e.* $0 < \underline{C}_\Theta \le \overline{C}_\Theta \le c < \infty$*, and a finite time moment* $\overline{t}_\Theta \in \{1, 2, \ldots\}$*, such that* $\forall t > \overline{t}_\Theta,\ \forall \mathcal{K}_t \in \mathcal{K}_t{}^\star$

$$\underline{C}_\Theta \mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}) \le {}^o\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1}) \le \overline{C}_\Theta \mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}). \quad (60)$$

Then, the posterior pd $\mathsf{f}(\Theta|\mathcal{K}_{t-1})$ *(54) converges almost surely (with respect to* ${}^o\mathsf{f}$ *to a pd* $\mathsf{f}(\Theta|\mathcal{K}_\infty)$*. Its support coincides with the set of minimising arguments in*

$$\mathrm{supp}\left[\mathsf{f}(\Theta|\mathcal{K}_\infty)\right] = \mathrm{Arg} \min_{\Theta \in \mathrm{supp}[\mathsf{f}(\Theta)] \cap \Theta^\star} \mathsf{D}_\infty \left({}^o\mathsf{f} || \Theta \right). \quad (61)$$

Proof I

Proof Under the natural conditions of DM (45), the posterior pd (54) reads

$$
\begin{aligned}
\mathsf{f}(\Theta|\mathcal{K}_t) &\propto \mathsf{f}(\Theta)\exp[-t\mathsf{D}(\mathcal{K}_t,\Theta)], && (62)\\
\mathsf{D}(\mathcal{K}_t,\Theta) &= \frac{1}{t}\sum_{\tau\leq t}\ln[\eta(\mathcal{K}_\tau,\Theta)] && (63)\\
\eta(\mathcal{K}_\tau,\Theta) &\equiv \frac{{}^{o}\mathsf{f}(\Delta_\tau|A_\tau,\mathcal{K}_{\tau-1})}{\mathsf{f}(\Delta_\tau|\Theta,A_\tau,\mathcal{K}_{\tau-1})}.
\end{aligned}
$$

This form exploits that the non-normalised posterior pd can be multiplied by a Θ-independent factor. Let us fix the argument $\Theta\in\Theta^\star$ and set

$$
\begin{aligned}
& e_{\tau;\Theta} \equiv \ln(\eta(\mathcal{K}_\tau,\Theta)) - {}^{o}\mathsf{E}\left[\ln(\eta(\mathcal{K}_\tau,\Theta))\middle|\, A_\tau,\mathcal{K}_{\tau-1}\right]\\
& \equiv \ln(\eta(\mathcal{K}_\tau,\Theta)) - \int_{\Delta_\tau{}^\star} {}^{o}\mathsf{f}(\Delta_\tau|A_\tau,\mathcal{K}_{\tau-1})\ln(\eta(\mathcal{K}_\tau,\Theta))\,\mathrm{d}\Delta_\tau.
\end{aligned}
$$

Proof II

- A direct check reveals that the deviations $e_{\tau;\Theta}$ of $\ln(\eta(\mathcal{K}_\tau, \Theta))$ from their conditional expectations ${}^o\mathsf{E}[\ln(\eta(\mathcal{K}_\tau, \Theta))|A_\tau, \mathcal{K}_{\tau-1}]$, given by ${}^o\mathsf{f}(\Delta_\tau|A_\tau, \mathcal{K}_{\tau-1})$, are zero mean and mutually uncorrelated. With them,
$$\mathsf{D}(\mathcal{K}_t, \Theta) = \mathsf{D}_t\left({}^o\mathsf{f}||\Theta\right) + \frac{1}{t}\sum_{\tau \le t} e_{\tau;\Theta}.$$
- The first right-hand term is nonnegative, Proposition 17. Due to (60), it is also finite and converges for $t \to \infty$.
- Assumption (60) implies that variance of $e_{\tau;\Theta}$ is bounded. Thus, the second sum converges to zero a.s.; see [Loe62], page 417, and (63) converges a.s. $\mathsf{D}_\infty\left({}^o\mathsf{f}||\Theta\right) \ge 0$.
- The posterior pd remains unchanged if we subtract $t \min_{\Theta \in \mathrm{supp}[\mathsf{f}(\Theta)] \cap \Theta^\star} \mathsf{D}_\infty\left({}^o\mathsf{f}||\Theta\right)$ in the exponent of of (62). Then, the exponent contains ($-t\times$ an asymptotically nonnegative factor) and the pd $\mathsf{f}(\Theta|\mathcal{K}_\infty)$ may be nonzero on the set (61) only.

□

Remark 12

- *The entropy rate↰ extends the KLD↰ (58). It covers asymptotic and controlled cases and often coincides with the KLD.*
- *Assumption (60) excludes a parametric model↰ that does not expect some observations generated by the system↰ with nonzero probability and vice versa.*
- *The Bayesian estimation asymptotically finds*
- *the best projection , which is the minimiser in the set of parametric models* $\{(\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}))_{t\in t^\star}\}_{\Theta\in\Theta^\star}$ *of the entropy rate↰ of the objective pd↰* ${}^{o}\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1})$ *on it.*
- *The prior pd↰ assigns prior belief to* $\Theta \in \Theta^\star$ *that the parametric model↰ is the best projection of the objective pd↰ [BK97]: not knowing it we do not know the best projection.*
- *The model is identifiable if the entropy rate↰ has a unique minimiser. Identifiability depends on i) the parametric-models' set; ii) the actions used (e.g. zero inputs hide their dynamic influence on outputs).*

Remark 13

- *If the objective pd* $^{o}\mathsf{f}(\Delta_t|A_t,\mathcal{K}_{t-1})$ *coincides with* $\mathsf{f}(\Delta_t|\Theta,A_t,\mathcal{K}_{t-1})$ *for some* $\Theta = {}^{o}\Theta \in \Theta^{\star}$ *with* $\mathsf{f}\,({}^{o}\Theta) > 0$ *then* ${}^{o}\Theta$ *is in the support of the asymptotic posterior pd* $\mathsf{f}(\Theta|\mathcal{K}_{\infty})$*. If, moreover, the model is identifiable, then the objective pd is asymptotically identified by the adopted Bayesian approach. This has the appealing expression:*

  Bayesian estimator is consistent whenever there is a consistent one.

- *Often, a similar analysis is performed by measuring the distance of the parametric model to the empirical pd of data [San57]. It gives similar answers if the empirical pd converges to the objective pd. Moreover, it provides hints of how to approximate the posterior pd [Kul96]; see also Section 27. On the other hand, the known conditions of such convergence are more restrictive. For instance, the analysis of the estimation in closed decision loop is much harder.*

Fully Probabilistic DM Design

This part goes beyond the traditional design↰. It justifies the fully probabilistic design (FPD↰) of DM↰ strategies and finds its position with respect to the traditional Bayesian DM↰. It summaries, unifies and complements results described in [K9́6, GK05, KG06, ŠVK08, Kár07].

Representative of Strategy Ordering $\preccurlyeq_{\mathsf{S}^\star}$

- The found functional representing the complete ordering of strategies $\preccurlyeq_{\mathsf{S}^\star}$, (14), is

$$
\begin{aligned}
\mathsf{E}_\mathsf{S}[\mathsf{I}_\mathsf{S}] &= \int_{B^\star} \mathsf{I}_\mathsf{S}(B)\mathsf{f}_\mathsf{S}(B)\,\mathrm{d}B, \quad \text{where, see (19),} \\
\mathsf{I}_\mathsf{S}(B) &= \mathsf{U}(\mathsf{Z}(B), \mathsf{W}^{-1}(\mathsf{S}, B)) \quad \text{with, see (2),} \\
B &= \mathsf{W}(\mathsf{S}, N), \qquad N \in N^\star \neq \emptyset.
\end{aligned}
$$

- The traditional design does not consider dependence of the performance index I_S on the strategy S.
- Here, this dependence is modelled under an additional, widely acceptable, assumption on the utility U defining the performance index I_S.

Ideal Pd

- The chosen performance index↰ I serves for the design↰ of the optimal strategy↰

$$ {}^{O\mathsf{I}}\mathsf{S} \equiv {}^{O}\mathsf{S} \in \mathrm{Arg}\min_{\mathsf{S}\in\mathsf{S}^\star} \int_{B^\star} \mathsf{I}(B, \mathsf{W}^{-1}(\mathsf{S}, B))\mathsf{f}_\mathsf{S}(B)\,\mathrm{d}B. \tag{64} $$

The corresponding closed loop model leads to the important notion:

- *ideal pd* ${}^{\mathsf{I}}\mathsf{f}(B)$ is interpreted as the pd↰ describing the behaviour of the decision loop closed by the optimal strategy↰ ${}^{O\mathsf{I}}\mathsf{S}$ (64), i.e. ${}^{\mathsf{I}}\mathsf{f}(B) \equiv \mathsf{f}_{{}^{O\mathsf{I}}\mathsf{S}}(B)$.
- All performance indices leading to optimal strategies that provide the same ideal pd, the same closed loop model, are obviously equivalent from DM design↰ view point.
- Further on, we inspect the DM design↰ that – instead of the specification of equivalent performance indices – starts with the specification of the ideal pd↰.

target

Utility Models Risk Attitude of Decision Makers

Further on, we consider utilities modelling properly the risk attitude.

Requirement 3 (Risk Respecting Utility)

The utility U *respects risk attitude properly if it meets the implication*

$$\begin{aligned} & \mathsf{f_S}({}^{a}B) = \mathsf{f_S}({}^{b}B) \text{ for } {}^{a}B,\ {}^{b}B \in B^{\star} \\ \Rightarrow\ & \mathsf{U}(z, \mathsf{W}^{-1}(\mathsf{S}, {}^{a}B)) = \mathsf{U}(z, \mathsf{W}^{-1}(\mathsf{S}, {}^{b}B)),\ \forall z \in (-\infty, \infty). \end{aligned} \tag{65}$$

- Requirement 3 means that two equally probable behaviours ${}^{a}B$, ${}^{b}B$ leading to the same value z of the loss contribute to the expected performance index↰ equally.
- Under Requirement 3, the performance index↰ has to have form

$$\mathsf{I_S}(B) \equiv \mathsf{I}(B, \mathsf{W}^{-1}(\mathsf{S}, B)) = \mathsf{I}(B, \mathsf{f_S}(B)). \tag{66}$$

Representative of Performance Indices with Given Ideal Pd

Here we assume that the decision maker has provided an ideal pd↰ ${}^{\mathsf{I}}\mathsf{f}(B)$ instead of a performance index↰. We search for a representative of the equivalence class of performance indices having the given ideal pd↰ as the closed loop model↰ with optimal strategy.

Requirement 4 (On Representative Performance Index)

The performance index↰

- *meets Requirement 3, i.e. it has the form* $\mathsf{I}(B, \mathsf{f}_{\mathsf{S}}(B))$;
- *has continuous derivative with respect to the second argument for almost all* $B \in B^{\star}$;
- *guarantees*

$$\begin{aligned} {}^{o}\mathsf{S} &\in Arg\min_{S \in S^{\star}} \int_{B^{\star}} \mathsf{I}(B, \mathsf{f}_{\mathsf{S}}(B))\, \mathrm{d}B \\ {}^{\mathsf{I}}\mathsf{f}(B) &= \mathsf{f}_{{}^{o}\mathsf{S}}(B) \end{aligned} \tag{67}$$

- *fulfills*

$$\mathsf{I}(B, {}^{\mathsf{I}}\mathsf{f}(B)) = constant, \ \forall B \in B^{\star}. \tag{68}$$

FPD Represents DM Tasks with Common Ideal Pd

Proposition 19 (Representative of Performance Indices Sharing Ideal)

For a given ideal pd↰ ${}^{\mathsf{I}}\mathsf{f}(B) > 0$ on $B^\star$, the representative of the performance indices sharing it, which meets Requirement 4, exists and up to linear transformation has the form

$$\mathsf{I}(B, \mathsf{f}_S(B)) \;=\; \ln\left(\frac{\mathsf{f}_S(B)}{{}^{\mathsf{I}}\mathsf{f}(B)}\right), \textit{i.e.} \tag{69}$$

the expected performance index↰ to be minimised is the KLD↰

$$\mathsf{E}_S[\mathsf{I}_S] \;=\; \mathsf{D}(\mathsf{f}_S||\,{}^{\mathsf{I}}\mathsf{f}) = \int_{B\in B^\star} \mathsf{f}_S(B)\ln\left(\frac{\mathsf{f}_S(B)}{{}^{\mathsf{I}}\mathsf{f}(B)}\right) \mathrm{d}B. \tag{70}$$

- *FPD* , fully probabilistic design of DM↰ strategy↰, is the optimal design↰ with the performance index↰ (69). It takes

$${}^{O}\mathsf{S} \in \mathrm{Arg}\min_{\mathsf{S}\in\mathsf{S}^\star} \mathsf{D}(\mathsf{f}_S||\,{}^{\mathsf{I}}\mathsf{f}) \;\text{ as the optimal strategy↰.} \tag{71}$$

Proof of the Form of Representative of Performance Indices Sharing Ideal Pd

Proof Under Requirement 4, (67) implies that the variation of the minimised functional has to vanish for $\mathsf{f}_\mathsf{S}(B) = {}^\mathsf{I}\mathsf{f}(B)$. This gives the necessary condition for almost all $B \in B^\star$

$$x \frac{\partial}{\partial x} \mathsf{I}(B, x) + \mathsf{I}(B, x) = \mathit{constant}, \text{ for } x = {}^\mathsf{I}\mathsf{f}(B) > 0. \tag{72}$$

Under Requirement 4, the identity (68) implies that (72) has the solution

$$\mathsf{I}(B, \mathsf{f}_\mathsf{S}(B)) = \mathit{constant} \times \ln\left(\frac{\mathsf{f}_\mathsf{S}(B)}{{}^\mathsf{I}\mathsf{f}(B)}\right) + \mathit{Constant}.$$

The minimum is reached for the *constant* > 0 only. □

Data-Driven FPD: Formalisation

- To get a feeling about this non-standard design, we start with data-driven design↰ with behaviour $B = D^h$ = sequence of data record↰s $= (A^h, \Delta^h)$ = sequence of (action↰s,observation↰s) pairs.
- In this case, the joint pd↰ $\mathsf{f}(B) \equiv \mathsf{f}(D^h)$ factorises by a repetitive use of the chain rule↰

$$\mathsf{f}(D^h) = \prod_{t \in t^\star} \mathsf{f}(\Delta_t | A_t, D^{t-1}) \mathsf{f}(A_t | D^{t-1}). \tag{73}$$

We consider FPD↰ determined an ideal pd↰ factorised similarly

$$^{\mathsf{I}}\mathsf{f}(D^h) = \prod_{t \in t^\star} {}^{\mathsf{I}}\mathsf{f}(\Delta_t | A_t, D^{t-1}) \, {}^{\mathsf{I}}\mathsf{f}(A_t | D^{t-1}) \tag{74}$$

and the optimal strategy minimising the KLD↰ (58) of $\mathsf{f}(D^h)$ on ${}^{\mathsf{I}}\mathsf{f}(D^h)$.

Data-Driven FPD: Solution

Proposition 20 (Solution of Data-Driven FPD)

The randomised decision rules of the optimal strategy in the data-driven FPD are

ofAtex1
ofAtex2

$$
\begin{aligned}
{}^{O}\mathsf{f}(A_t|D^{t-1}) &= {}^{I}\mathsf{f}(A_t|D^{t-1}) \frac{\exp[-\omega_\gamma(A_t, D^{t-1})]}{\gamma(D^{t-1})} && (75) \\
\gamma(D^{t-1}) &\equiv \int_{A_t^\star} {}^{I}\mathsf{f}(A_t|D^{t-1}) \exp[-\omega(A_t, D^{t-1})] \, \mathrm{d}A_t \leq 1 && (76) \\
& \quad \text{for } t < h \text{ and } \gamma(D^h) = 1 && (77) \\
\omega_\gamma(A_t, D^{t-1}) &\equiv \int_{\Delta_t^\star} \mathsf{f}(\Delta_t|A_t, D^{t-1}) \ln \left(\frac{\mathsf{f}(\Delta_t|A_t, D^{t-1})}{\gamma(D^t)\, {}^{I}\mathsf{f}(\Delta_t|A_t, D^{t-1})} \right) d\Delta_t. && (78)
\end{aligned}
$$

The solution is performed against the time course, starting at $t = h$.

Data-Driven FPD: Proof I

Proof The product forms of the closed loop model (73) and its ideal counterpart (74) imply that KLD is additive form with the partial performance index $\mathsf{z}(D^t)$

$$= \ln\left(\frac{\mathsf{f}(\Delta_t|A_t,\mathcal{K}_{t-1})\mathsf{f}(A_t|\mathcal{K}_{t-1})}{{}^{\mathsf{I}}\mathsf{f}(\Delta_t|A_t,\mathcal{K}_{t-1})\,{}^{\mathsf{I}}\mathsf{f}(A_t|\mathcal{K}_{t-1})}\right) = \ln\left(\frac{\mathsf{f}(\Delta_t|A_t,D^{t-1})\mathsf{f}(A_t|D^{t-1})}{{}^{\mathsf{I}}\mathsf{f}(\Delta_t|A_t,D^{t-1})\,{}^{\mathsf{I}}\mathsf{f}(A_t|D^{t-1})}\right).$$

Thus, we face a variation of Proposition 12. Let us express loss-to-go in the form $-\ln(\gamma(D^t))$. It defines the terminal condition $\gamma(D^h) = 1$ and the generic term to be minimised over $\mathsf{f}(A_t|D^{t-1})$ reads

$$\int_{D_t^\star} \mathsf{f}(\Delta_t|A_t,D^{t-1})\mathsf{f}(A_t|D^{t-1}) \ln\left(\frac{\mathsf{f}(\Delta_t|A_t,D^{t-1})\mathsf{f}(A_t|D^{t-1})}{\gamma(D^t)\,{}^{\mathsf{I}}\mathsf{f}(\Delta_t|A_t,D^{t-1})\,{}^{\mathsf{I}}\mathsf{f}(A_t|D^{t-1})}\right) \mathrm{d}D_t$$

$$= \int_{A_t^\star} \mathsf{f}(A_t|D^{t-1}) \left[\ln\left(\frac{\mathsf{f}(A_t|D^{t-1})}{{}^{\mathsf{I}}\mathsf{f}(A_t|D^{t-1})}\right) + \underbrace{\omega(A_t,D^{t-1})}_{(78)}\right] \mathrm{d}A_t$$

Data-Driven FPD: Proof II

$$= \; -\ln\left(\int_{A_t^\star} {}^{\mathsf{I}}\mathsf{f}(A_t|D^{t-1})\exp\left[-\omega(A_t, D^{t-1})\right]\, \mathrm{d}A_t\right)$$

$$+ \; \int_{A_t^\star} \mathsf{f}(A_t|D^{t-1}) \ln\left(\frac{\mathsf{f}(A_t|D^{t-1})}{\frac{{}^{\mathsf{I}}\mathsf{f}(A_t|D^{t-1})\exp[-\omega(A_t,D^{t-1})]}{\int_{A_t^\star} {}^{\mathsf{I}}\mathsf{f}(A_t|D^{t-1})\exp[-\omega(A_t,D^{t-1})]\,\mathrm{d}A_t}}\right)\mathrm{d}A_t.$$

The last term depends on the optimised decision rule $\mathsf{f}(A_t|D^{t-1})$ and it is conditional version of KLD of $\mathsf{f}(A_t|D^{t-1})$ on the decision rule (75). Proposition 17 implies that minimiser has to have the form (75) and the first term above is the minimum reached. It defines the loss-to-go for the subsequent optimisation step. □

Remark 14

- *For an alternative derivations [K\'96, vK97, ŠVK08].*
- *At a descriptive level, the dynamic programming, Proposition 11, consists of the evaluation pairs*

$$(\textit{conditional expectation, minimisation}).$$

Except of a few numerically solvable cases, some approximation technique is needed. The complexity of the approximated optimum prevents a systematic use of the standard approximation theory.

- *Systematic approximations [Ber01, SBPW04] are still not matured enough and various ad hoc schemes are adopted. The FPD↰ finds minimisers explicitly and reduces the design↰ to a sequence of conceptually feasible multivariate integrations.*
- *The found optimal strategy↰ is a randomised and causal one. The physical constraints↰ are met trivially if the chosen ideal strategy respects them, i.e. if* $\mathrm{supp}\left[\,{}^{\mathsf{I}}\mathsf{f}(A_t|\mathcal{K}_{t-1})\right] \subset A_t^{\star}$, *cf. (75).*

FPD and Traditional Design: Aim of Exposition

- The fully probabilistic design forms the core of our view on DM.
- Here, it is shown that traditional DM design↰s form a proper subset of DM↰ formulated as FPD↰.
- Thus, no DM task is neglected.

Discrepancy in the Interpretation of Ideal Pd

- Dynamic programming, Proposition 11, that solves the traditional DM design↰ leads to the optimal deterministic strategy.
- The ideal pd↰ is interpreted as closed loop model↰ with the optimal strategy↰.

$$\Downarrow$$

- The positivity of the ideal pd↰ required in Proposition 19 leading to FPD is violated.

The following technical tool helps us in coping with the discrepancy.

Proposition 21 (Lower Bound on Entropy of Deterministic Rules)

Any deterministic rule $\mathsf{f}(A|\mathcal{K}_{A^\star}) = \delta(A - A(\mathcal{K}_{A^\star}))$*, where Dirac delta*↰ *concentrates on a point* $A(\mathcal{K}_{A^\star}) \in A^\star$*, reaches the lower bound* $\underline{H}$ *of the*

- *entropy* $\mathsf{H}(\mathsf{f}) = -\int_{A^\star} \mathsf{f}(A) \ln(\mathsf{f}(A))\, \mathrm{d}A$.

$$\underline{H} = \begin{cases} 0 & \textit{for discrete-valued action } A \\ -\infty & \textit{continuous-valued action } A \end{cases}. \tag{79}$$

Proof Direct inspection solves discrete-valued case. In continuous-valued case, the Dirac delta↰ is generalised function, which can be obtained as a limit of positive pds [Vla79], say normal ones with the mean $A(\mathcal{K}_{A^\star})$ and diagonal covariance with diagonal entry $\varepsilon > 0$ approaching to zero. For such pds the entropy↰ equals $0.5\ell_A \ln(2\pi\varepsilon) \to -\infty$ for $\varepsilon \to 0$. □

Traditional Design with Constrained Entropy

Proposition 22 (Properties of Entropy-Constrained Traditional DM)

Let a system model↰ M *be given and the traditional DM design*↰ *with a performance index*↰ $\mathsf{I}(B)$ *be solved. The admissible strategy*↰ S *minimising the expected performance index under the constraint*

$$\int_{B^\star} \mathsf{f}_\mathsf{S}(B)\ln(\mathsf{S}(B))\,\mathrm{d}B = \int_{B^\star} \mathsf{M}(B)\mathsf{S}(B)\ln(\mathsf{S}(B))\,\mathrm{d}B \le \bar{H} < -\underline{H}, \quad (80)$$

attains the constraint (80), and approaches the optimal Bayesian strategy when $\bar{H} \to -\underline{H}$.

Proof It is a direct consequence of Propositions 11 and 21. □

Entropy-Constrained Traditional DM

Proposition 23 (FPD Is Entropy-Constrained Traditional DM)

The traditional DM design with the entropy constrained by $\bar{H} < -\bar{H}$ (80) $\Leftrightarrow$ FPD with determined by

$$\textit{the ideal pd } {}^{\mathsf{I}\bar{H}}\mathsf{f}(B) = \frac{\mathsf{M}(B)\exp\left[-\frac{\mathsf{I}(B)}{\lambda(\bar{H})}\right]}{\int_{B^\star}\mathsf{M}(B)\exp\left[-\frac{\mathsf{I}(B)}{\lambda(\bar{H})}\right]\,\mathrm{d}B} > 0. \quad (81)$$

The multiplier $\lambda(\bar{H}) > 0$ goes to zero for $\bar{H} \to -\underline{H}$ and the optimal strategy converges to the optimal strategy of the traditional DM design.

Proof Kuhn-Tucker theorem [KT51] implies that the task reduces to

$$\begin{aligned} {}^{\mathsf{I}\bar{H}}\mathsf{S} &\in \operatorname*{Arg\,min}_{\mathsf{S}\in\mathsf{S}^\star}\int_{B^\star}\left[\mathsf{I}(B) + \lambda(\bar{H})\ln(\mathsf{S})\right]\mathsf{f}_{\mathsf{S}}(B)\,\mathrm{d}B \\ &= \operatorname*{Arg\,min}_{\mathsf{S}\in\mathsf{S}^\star}\mathsf{D}(\mathsf{f}_{\mathsf{S}}\|\,{}^{\mathsf{I}\bar{H}}\mathsf{f}) \;\text{ with }\; {}^{\mathsf{I}\bar{H}}\mathsf{f}(B)\ (81). \end{aligned}$$

Limiting properties are implied by Proposition 22 describing the behaviour for a gradually relaxed constraint (80). □

FPD Is Proper Dense Extension of Traditional DM

A generic optimal strategy obtained by FPD↰ are randomised, see Proposition 20. Thus, it holds.

Proposition 24 (FPD vs. Standard Bayesian DM)

- *Any traditional DM design↰ can be approximated to an arbitrary precision by the FPD problem with the ideal pd (81) when selecting sufficiently small positive $\lambda(\bar{H})$.*
- *There are FPD's having no standard counterpart.*

On Interpretations of Constrained Entropy

Remark 15

- *The constraint (80) is connected with Agreement 1. The optimal strategy↰ is to be implemented and actions transmitted through a real interface. At least for real-valued actions, no communication channel (computer, sensor, actuator) transmits them without distortion: no deterministic strategy is exactly implementable.*
- *The constraint (80) can be related to constraints on computational complexity or deliberation effort [Per55], to "rational inattention" [Sim02] or to a numerical solution with Boltzman machine [SBPW04].*
- *The presented ideal pd helps practically: designers are trained to construct a loss↰ quantifying the design aim↰. The formula (81) decreases the natural barrier to FPD↰. Moreover, it enables the use of data for correction of the ideal pd↰,i.e. the data-dependent estimation of preferential ordering↰.*

The most general FPD↰ operates on the DM elements.

- *DM elements* are specified for discrete time $t \in t^\star$ up to decision horizon↰ $h < \infty$. The behaviour↰ $B = (X^h, D^h) = (X^h, \Delta^h, A^h) =$ (hidden,data record↰) = (hiddens,action↰,observation↰) belongs to set $B^\star = (X^{h\star}, D^{h\star}) = (X^{h\star}, A^{h\star}, \Delta^{h\star})$. They consist of
  - admissible strategy↰ given by decision rule↰s with pds meeting natural conditions of DM↰ $\mathsf{f}(A_t|X^{t-1}, \mathcal{K}_{t-1}) = \mathsf{f}(A_t|\mathcal{K}_{t-1})$
  - data record↰s $D_t = (A_t, \Delta_t) =$ (action↰,observation↰)$\in D_t^\star = \mathcal{K}_t{}^\star - \mathcal{K}_{t-1}^\star$
  - observation model↰ $\mathsf{f}(\Delta_t|X_t, A_t, \mathcal{K}_{t-1})$
  - time evolution model↰ $\mathsf{f}(X_t|X_{t-1}, A_t, \mathcal{K}_{t-1})$
  - prior pd↰ $\mathsf{f}(X_0, \mathcal{K}_0|A_1) = \mathsf{f}(X_0|\mathcal{K}_0)\mathsf{f}(\mathcal{K}_0)$ reflecting prior knowledge, cf. (45)
  - ideal strategy ${}^{\mathsf{I}}\mathsf{f}(A_t|X^{t-1}, \mathcal{K}_{t-1})$
  - ideal observation model ${}^{\mathsf{I}}\mathsf{f}(\Delta_t|X_t, A_t, \mathcal{K}_{t-1})$
  - ideal time evolution model ${}^{\mathsf{I}}\mathsf{f}(X_t|X_{t-1}, A_t, \mathcal{K}_{t-1})$
  - ideal prior pd ${}^{\mathsf{I}}\mathsf{f}(X_0, \mathcal{K}_0|A_1)$.
- Proposition 14 provides the predictor $\mathsf{f}(\Delta|A_t, \mathcal{K}_{t-1})$ and the pd $\mathsf{f}(X_{t-1}|\mathcal{K}_{t-1})$.

Proposition 25 (Solution of General FPD)

The strategy solving FPD with general DM elements has the rules, $t \in t^\star$,

$$
{}^{O}\mathsf{f}(A_t|\mathcal{K}_{t-1}) = \frac{\exp[-\omega(A_t,\mathcal{K}_{t-1})]}{\gamma(\mathcal{K}_{t-1})}, \ \gamma(\mathcal{K}_{t-1}) \equiv \int_{A_t^\star} \exp[-\omega(A_t,\mathcal{K}_{t-1})]\,\mathrm{d}A_t
$$

$$
\begin{aligned}
\omega(A_t,\mathcal{K}_{t-1}) = \ & - \int_{\Delta_t^\star} \mathsf{f}(\Delta_t|A_t,\mathcal{K}_{t-1})\,\mathrm{d}\Delta_t \ln(\gamma(\mathcal{K}_t)) \qquad (82)\\
& - \int_{X_{t-1}^\star} \mathsf{f}(X_{t-1}|\mathcal{K}_{t-1})\,\mathrm{d}X_{t-1} \left\{ \ln({}^{I}\mathsf{f}(A_t|X_{t-1},\mathcal{K}_{t-1})) \right.\\
& + \int_{X_t^\star} \mathsf{f}(X_t|X_{t-1}A_t,\mathcal{K}_{t-1})\,\mathrm{d}X_t \left[\ln\left(\frac{\mathsf{f}(X_t|X_{t-1},A_t,\mathcal{K}_{t-1})}{{}^{I}\mathsf{f}(X_t|X_{t-1},A_t,\mathcal{K}_{t-1})} \right) \right.\\
& \left.\left. + \int_{\Delta_t^\star} \mathsf{f}(\Delta_t|X_t,A_t,\mathcal{K}_{t-1})\,\mathrm{d}\Delta_t \ \ln\left(\frac{\mathsf{f}(\Delta_t|X_t,A_t,\mathcal{K}_{t-1})}{{}^{I}\mathsf{f}(\Delta_t|X_t,A_t,\mathcal{K}_{t-1})} \right) \right] \right\}.
\end{aligned}
$$

The recursion starts with $\gamma(\mathcal{K}_h) \equiv 1$ *and runs backwards.*

Proof Let us define the loss-to-go↰ corresponding to FPD

$$-\ln(\gamma(\mathcal{K}_{t-1})) = \min_{\{\mathsf{f}(A_\tau|\mathcal{K}_{\tau-1})\}_{\tau=t}^{h}} \sum_{\tau=t}^{h} \int_{\Delta_\tau^\star, A_\tau^\star, X_\tau^\star, X_{\tau-1}^\star} \mathsf{f}(\Delta_\tau, A_\tau, X_\tau, X_{\tau-1}|\mathcal{K}_{\tau-1})$$

$$\times \quad \ln\left(\frac{\mathsf{f}(\Delta_\tau, A_\tau, X_\tau, X_{\tau-1}|\mathcal{K}_{\tau-1})}{{}^{\mathsf{I}}\mathsf{f}(\Delta_\tau, A_\tau, X_\tau, X_{\tau-1}|\mathcal{K}_{\tau-1})}\right) \mathrm{d}\Delta_\tau\, \mathrm{d}A_\tau\, \mathrm{d}X_\tau\, \mathrm{d}X_{\tau-1}.$$

The form of KLD↰ implies that $\gamma(\mathcal{K}_0)$ coincides with its minimum and minimising decision rule↰s form the optimal strategy in FPD sense and

$$-\ln(\gamma(\mathcal{K}_{t-1})) = \min_{\mathsf{f}(A_t|\mathcal{K}_{t-1})} \int_{A_t^\star} \mathsf{f}(A_t|K_t) \ln\left(\frac{\mathsf{f}(A_t|\mathcal{K}_{t-1})}{\exp(-\omega(A_t, \mathcal{K}_{t-1}))}\right) \mathrm{d}A_t$$

$$\omega(A_t, \mathcal{K}_{t-1}) = \int_{\Delta_\tau^\star, X_\tau^\star, X_{\tau-1}^\star} \mathsf{f}(\Delta_\tau|A_\tau, X_\tau, \mathcal{K}_{\tau-1})\mathsf{f}(X_\tau|A_\tau, X_{\tau-1}, \mathcal{K}_{\tau-1}) \ln\Bigg($$

$$\frac{\mathsf{f}(\Delta_\tau|A_\tau, X_\tau, \mathcal{K}_{\tau-1})\mathsf{f}(X_\tau|A_\tau, X_{\tau-1}, \mathcal{K}_{\tau-1})\mathsf{f}(X_{\tau-1}|\mathcal{K}_{\tau-1})}{\gamma(\mathcal{K}_t)\,{}^{\mathsf{I}}\mathsf{f}(\Delta_\tau|A_\tau, X_\tau, \mathcal{K}_{\tau-1})\,{}^{\mathsf{I}}\mathsf{f}(X_\tau|A_\tau, X_{\tau-1}, \mathcal{K}_{\tau-1})\,{}^{\mathsf{I}}\mathsf{f}(A_t|X_{t-1}, \mathcal{K}_{t-1})\,{}^{\mathsf{I}}\mathsf{f}(X_{\tau-1}|\mathcal{K}_{\tau-1})}\Bigg)$$

$$\times \quad \mathsf{f}(X_{\tau-1}|\mathcal{K}_{\tau-1})\, \mathrm{d}\Delta_\tau\, \mathrm{d}X_\tau\, \mathrm{d}X_{\tau-1}$$

This and (82) exploit marginalisation↰, normalisation↰, chain rule↰, natural conditions of DM↰, Fubini theorem [Rao87b] and KLD↰ minimiser. □

Constructing of Design Elements

- This part provides a (still incomplete) methodology of quantitative construction of DM elements↰ from elements that are expected to be available practically.
- Primarily, the use of FPD↰, that covers traditional DM design↰, assumes ability to specify:
  - system model↰ given by the pd M (50);
  - the ideal pd↰ ${}^{\mathsf{I}}\mathsf{f} = {}^{\mathsf{I}}\mathsf{M}\,{}^{\mathsf{I}}\mathsf{S}$ specifying DM preferences, constraints and risk attitude.

  They are discussed here.
- The discussion of the inevitable specification of
  - admissible strategies $\mathsf{S}^\star$ among which the optimal strategy↰ ${}^{O}\mathsf{S}$ (71) is searched for;
  - knowledge $K^\star$, data $D^\star$, hidden' $H^\star$ and actions' $A^\star$ sets on which the involved functions act;

  as well as abilities to evaluate the strategy ${}^{O}\mathsf{S}$, i.e. store, integrate and optimise functions in Proposition 25 and finally apply ${}^{O}\mathsf{S}$, are postponed.

Solution Concept and Local Notation

- Approximation of pds and extension of a partially specified pd are basic tools for practical construction of DM elements.
- These problems are formulated as specific supporting DM tasks. They are also formulated and solved as the FPD.
- The DM constituents of the supporting DM task are denoted by calligraphic counterparts of symbols used in the supported DM task in order easily recognise DM elements of the supporting DM task.
- Technicalities connected with infinite-dimensional random variables are avoided by assuming the behaviour $B \in B^\star$ (of the supported DM) to have a finite number of instances $|B^\star| < \infty$. Consequently, the inspected pds f of the supported DM

$$\mathsf{f} \in \mathsf{f}^\star \subset \mathsf{f}^\star_\triangle = \left\{ \mathsf{f}(B) : \mathsf{f}(B) \geq 0, \int_{B^\star} \mathsf{f}(B)\,\mathrm{d}B = 1 \right\} \tag{83}$$

are finite dimensional vectors.
- Pds like $\mathcal{F}(B, \mathsf{f})$ of the supporting DM thus make a good sense.

On Partially Specified Ideal Pd

- Decision maker has to specify the ideal closed loop model on whole behaviour B. Often, a part ${}^{i}B$ influences its preferential ordering and the rest ${}^{u}B$ serves as a knowledge source only.
- In the factorised ideal pd,

$$ {}^{I}f(B) = {}^{I}f({}^{u}B|{}^{i}B)\,{}^{I}f({}^{i}B) \tag{84} $$

  the decision maker is able or willing to specify ${}^{I}f({}^{i}B)$ only.
- In order to leave as much freedom to the design as possible, we must not enforce anything above the designer's (user's) wishes. Thus, we have to let the design decide on the distribution of quantities ${}^{u}B$ – we have to "leave them to their fate" [KBG+06],
- *leave to the fate* means that the ideal pd of $B = ({}^{u}B, {}^{i}B)$ has the form ${}^{I}f(B) = {}^{I}f({}^{u}B|{}^{i}B)\,{}^{I}f({}^{i}B) = f({}^{u}B|{}^{i}B)\,{}^{I}f({}^{i}B)$, where the pd $f({}^{u}B|{}^{i}B)$ results from the FPD with this ideal pd.

Approximation of Known Pd as FPD: Closed Loop Model

Often, a pd f constructed from the available knowledge is too complex to be treated by an imperfect decision makers and has to be approximated by a pd $\hat{\mathsf{f}} \in \hat{\mathsf{f}}^\star \subset \mathsf{f}^\star_\triangle$ (83). Approximation is a supporting DM problem with the following specification.

- The considered action $\mathcal{A} = \hat{\mathsf{f}}$ uses knowledge $\mathcal{K}_{\mathcal{A}^\star} = \mathsf{f} =$ the approximated pd and faces the ignorance $\mathcal{G}_{\mathcal{A}^\star} = B =$ the behaviour of the supported DM task.
- The behaviour of the supporting DM is $\mathcal{B} = (B, \hat{\mathsf{f}}, \mathsf{f})$ =(behaviour of the supported DM, its approximate pd, its approximated pd).
- The adopted system model $\mathcal{F}(\mathcal{G}_{\mathcal{A}^\star}|\mathcal{A}, \mathcal{K}_{\mathcal{A}^\star}) = \mathcal{F}(B|\hat{\mathsf{f}}, \mathsf{f}) = \mathsf{f}(B)$ uses the fact that the pd f models B. It combines with the optimised decision rule $\mathcal{S}(\hat{\mathsf{f}}|\mathsf{f})$ into the closed-loop model

$$\mathcal{F}(\mathcal{G}_{\mathcal{A}^\star}, \mathcal{A}|\mathcal{K}_{\mathcal{A}^\star}) = \mathsf{f}(B)\mathcal{S}(\hat{\mathsf{f}}|\mathsf{f}).$$

Approximation of Known Pd as FPD: Ideal Model

- The ideal closed-loop model is specified as

$$^{\mathsf{I}}\mathcal{F}(\mathcal{G}_{\mathcal{A}^\star}, \mathcal{A}|\mathcal{K}_{\mathcal{A}^\star}) = {}^{\mathsf{I}}\mathcal{F}(B, \hat{\mathsf{f}}|\mathsf{f}) = \hat{\mathsf{f}}(B)\mathcal{S}(\hat{\mathsf{f}}|\mathsf{f}).$$

- The choice of the first factor means that the approximating pd is to describe ideally the behaviour of the supported DM task.
- The choice of the second factor expresses a lack of additional requirements on the constructed decision rule $\mathsf{S}(\hat{\mathsf{f}}|\mathsf{f})$. The decision rule resulting from the design is accepted as the ideal one, the leave to the fate option is used.

Approximation of Known Pd as FPD: Solution

- With the options made, KLD↰ (71) of $\mathcal{F}_{\mathcal{S}}$ on ${}^{\mathsf{I}}\mathcal{F}$ is linear in $\mathcal{S}(\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star}) = \mathcal{S}(\hat{\mathsf{f}}|\mathsf{f})$, i.e. the FPD solving the supporting approximation task becomes traditional DM design↰.
- Basic DM lemma, Proposition 10, implies that the optimal decision rule↰ solving the supporting approximation DM task is deterministic and selects the optimal action↰, i.e. the optimal approximation

$$ {}^{\mathcal{O}}\hat{\mathsf{f}} \in \operatorname{Arg}\min_{\hat{\mathsf{f}} \in \hat{\mathsf{f}}^\star} \mathsf{D}(\mathsf{f}||\hat{\mathsf{f}}). \tag{85} $$

- *approximation principle* is expressed by (85); its rather different justification is in [Ber79].

Approximation of Known Pd as FPD: Example

Example 8 (Approximation by Normal Pd)

Let a given pd $\mathsf{f}(B)$ *is to be approximated by a normal pd*

$$\hat{\mathsf{f}}(B) = \mathcal{N}_B(\bar{B}, C) = |2\pi C|^{-0.5} \exp[-0.5(B - \bar{B})' C^{-1}(B - \bar{B})], \qquad (86)$$

where the approximating pds $\hat{\mathsf{f}}$ *are parameterised by the expected value* $\bar{B}$ *and the positive definite covariance matrix* C*. The optimal approximation* ${}^{O}\hat{\mathsf{f}}(B) = \mathcal{N}_B({}^{O}\bar{B}, {}^{O}C)$ *in the sense (85) is*

$$\begin{aligned} ({}^{O}\bar{B}, {}^{O}C) &\in \operatorname{Arg}\min_{\bar{B}, C} \mathsf{D}(\mathsf{f}||\mathcal{N}_B(\bar{B}^{\star}, C^{\star})) \\ &= \operatorname{Arg}\min_{\bar{B}^{\star}, C^{\star}} [\ln|C| + \mathsf{E}_{\mathsf{f}}[(B - \bar{B})' C^{-1}(B - \bar{B})] \\ &= (\mathsf{E}_{\mathsf{f}}[B], \operatorname{cov}_{\mathsf{f}}(B)), \end{aligned} \qquad (87)$$

${}^{O}\bar{B}$, ${}^{O}C$ *coincide with the expectation and the covariance of the approximated pd: the approximating pd matches the moments.*

Approximation of Unknown Pd as FPD: Behaviour

Unlike the approximation discussed above, here, the knowledge about the approximated pd is now more vague and the supporting action is a priori randomised. The addressed supporting DM problem is specified as follows.

- The action $\mathcal{A} = \mathcal{F}(\mathsf{f}) \in \mathcal{F}^\star$ is a pd on the unknown pds f describing the behaviour B of the supported DM.
- The knowledge about the approximated pd f is incomplete

$$\begin{aligned} \mathcal{K}_{\mathcal{A}^\star} : \quad & \mathsf{f} \in \mathsf{f}^\star \subset \mathsf{f}^\star_\triangle, \quad \text{see (83)}, \\ \mathsf{f}_0 \in \mathsf{f}^\star_\triangle \quad & \text{is the best prior (possibly flat) guess of f.} \end{aligned} \tag{88}$$

- The ignorance $\mathcal{G}_{\mathcal{A}^\star} = (B, \mathsf{f}) =$ (behaviour, its pd) of the supported DM.
- The behaviour $\mathcal{B} = (\mathcal{G}_{\mathcal{A}^\star}, \mathcal{A}, \mathcal{K}_{\mathcal{A}^\star}) = ((B, \mathsf{f}), \mathcal{F}(\mathsf{f}), (\mathsf{f}_0, \mathsf{f}^\star)) =$ ((behaviour of supported DM, its pd f),pd of f,(prior guess of $\mathsf{f} \in \mathsf{f}^\star_\triangle$,$\mathsf{f}^\star$)).

- The considered system model↰

$$\mathcal{F}(\mathcal{G}_{\mathcal{A}^\star}|\mathcal{A},\mathcal{K}_{\mathcal{A}^\star}) = \mathcal{F}(B,\mathsf{f}|\mathcal{F}(\mathsf{f}),(\mathsf{f}_0,\mathsf{f}^\star)) = \mathsf{f}(B)\mathcal{F}(\mathsf{f})$$

  uses that the pd f models B and the action $\mathcal{F}(\mathsf{f})$ is the pd of $\mathsf{f} \in \mathsf{f}^\star$.

- The optimised randomised decision rule↰ $\mathcal{S}(\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star})$ completes the specification of the closed-loop model↰

$$\mathcal{F}(\mathcal{G}_{A^\star},\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star}) = \mathsf{f}(B)\mathcal{F}(\mathsf{f})\mathcal{S}(\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star})$$

- The ideal closed loop model↰ is specified as

$$^{\mathsf{I}}\mathcal{F}(\mathcal{G}_{\mathcal{A}^\star},\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star}) = \mathsf{f}_0(B)\mathcal{F}(\mathsf{f})\mathcal{S}(\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star}). \quad (89)$$

- The choice (89) says that $\mathsf{f}_0(B)$ is taken as the best available description of the behaviour↰ B and there are no additional requirements on the constructed action↰ $\mathcal{F}(\mathsf{f})$ and the decision rule $\mathcal{S}(\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star})$ generating it. The results of the design↰ are accepted as the ideal ones, the leave to the fate↰ option is used.

Minimum KLD Principle Provides Solution

- Due ro the leave to the fate option, the action and the decision rule enter the optimised KLD linearly. It implies that the optimal decision rule and the optimal action are deterministic with a full mass on

$$ {}^{O}\mathsf{f} \in \operatorname{Arg}\min_{\mathsf{f}\in\mathsf{f}^\star} \mathsf{D}(\mathsf{f}||\mathsf{f}_0). \tag{90} $$

- The result (90) coincides with
- *minimum KLD principle* recommends to complete knowledge expressed by the set $\mathsf{f}^\star$ and the prior guess f_0 according to (90). It reduces to the maximum entropy principle if f_0 is uniform pd. Both principles are axiomatically justified in [SJ80] for the set $\mathsf{f}^\star \subset \mathsf{f}^\star_\triangle$ specified by given values μ of linear functionals given by a vector kernel ϕ: $\int_{B^\star} \phi(B)\mathsf{f}(B)\,\mathrm{d}B = \mu$ on $\mathsf{f}^\star_\triangle$ (83).

Example 9 (Uniform Pd Maximises Entropy)

- *Let* $\mathsf{f}^\star = \mathsf{f}^\star_\triangle$ *(83), i.e. no constrain is put on* f*. Then, properties of KLD*↰*, Proposition 17, imply that the optimal pd* ${}^{O}\mathsf{f}$ *in (90) coincides with the prior guess* f_0 *(88).*
- *The maximum entropy principle*↰ *coincides with the minimum KLD principle*↰ *for uniform prior guess. This makes a bit "circular" conclusion that uniform pd maximises entropy.*

Examples Related to Minimum KLD Principle (cont.)

Example 10 (Exponential Pd)

Let $\mathsf{f}^\star \subset \mathsf{f}^\star_\triangle$ *(83) be specified by a given* $\bar{B} = \mathsf{E}_\mathsf{f}[B]$. *Then, the optimal pd* ${}^{O}\mathsf{f}$ *in (90) minimises Lagrangian*

$$ {}^{O}\mathsf{f} \in \operatorname{Arg}\min_{\mathsf{f}^\star_\triangle} \mathsf{D}(\mathsf{f}||\mathsf{f}_0) + \lambda' \mathsf{E}_\mathsf{f}[B] \tag{91}$$

$$= \operatorname{Arg}\min_{\mathsf{f}^\star_\triangle} \mathsf{D}\left(\mathsf{f}\middle|\middle|\frac{\mathsf{f}_0\exp(-\lambda' B)}{\int_{B^\star}\mathsf{f}_0\exp(-\lambda' B)\,\mathrm{d}B}\right) = \frac{\mathsf{f}_0\exp(-\lambda' B)}{\int_{B^\star}\mathsf{f}_0\exp(-\lambda' B)\,\mathrm{d}B}$$

with λ *solving*

$$\bar{B} = \int_{B^\star} B\frac{\mathsf{f}_0\exp(-\lambda' B)}{\int_{B^\star}\mathsf{f}_0\exp(-\lambda' B)\,\mathrm{d}B}\,\mathrm{d}B$$

If $B^\star = \{B \geq 0\}$ *and* f_0 *is uniform on it then* $\lambda_i = 1/\bar{B}_i$ *and* ${}^{O}\mathsf{f}$ *becomes exponential pd.*

Examples Related to Minimum KLD Principle (cont.)

Example 11 (Normal Pd)

- *Let* $\mathsf{f}^\star \subset \mathsf{f}^\star_\triangle$ *(83) be specified by a given* $\bar{B} = \mathsf{E}_\mathsf{f}[B]$, $C = \mathrm{cov}_\mathsf{f}(B)$. *Then, the optimal pd* ${}^{O}\mathsf{f}$ *in (90) has the form*

$$ {}^{O}\mathsf{f} \propto \mathsf{f}_0 \exp[-0.5(B - \bar{\lambda})' \bar{C}^{-1}(B - \bar{\lambda})] \tag{92} $$

  with Lagrangian coefficient $\bar{\lambda}$, $\bar{C}$ *chosen to match moments* $\bar{B}$, C.

- *If* $B^\star$ *is* ℓ_B*-dimensional real space and* f_0 *is uniform on it then* $\bar{\lambda} = \bar{B}$ *and* $\bar{C} = C$, ${}^{O}\mathsf{f}$ *becomes normal pd* $\mathcal{N}_B(\bar{B}, C)$ *(86).*

Generalised Minimum KLD Principle

- The following alternative to the knowledge (88)

$$\mathcal{K}_{\mathcal{A}^\star} : \mathsf{f} \in \mathsf{f}^\star \subset \mathsf{f}^\star_\triangle,\ \mathcal{F}_0(\mathsf{f}) \in \mathcal{F}^\star, \text{ see (95)}, \tag{93}$$

  is the best available prior (flat) guess of the action $\mathcal{A} = \mathcal{F}(\mathsf{f})$

  changes just the ideal pd (89) to

$$^{\mathsf{I}}\mathcal{F}(B, \mathsf{f}, \mathcal{F}(\mathsf{f})|\mathcal{K}_{\mathcal{A}^\star}) = \mathsf{f}(B)\mathcal{F}_0(\mathsf{f})\mathcal{S}(\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star}). \tag{94}$$

  It respects that f models B, takes $\mathcal{F}_0(\mathsf{f})$ as the best prior guess of $\mathcal{A} = \mathcal{F}(\mathsf{f})$ and leaves the ignorance (B, f) and the decision rule $\mathcal{S}(\mathcal{A}|\mathcal{K}_{\mathcal{A}^\star})$ to their fate.
- The resulting choice of the deterministic decision rule generalises to

$$^{O}\mathcal{F} \in \mathrm{Arg} \min_{\mathcal{F} \in \mathcal{F}^\star} \int_{\mathsf{f}^\star} \mathcal{F}(\mathsf{f}) \ln\left(\frac{\mathcal{F}(\mathsf{f})}{\mathcal{F}_0(\mathsf{f})}\right) \mathrm{d}\mathsf{f} \tag{95}$$

$$\mathcal{F}^\star = \text{pds acting on } \mathsf{f}^\star, \text{ see (83)}.$$

- *generalised minimum KLD principle* is expressed by (95).

Use of Solutions of the Supporting DM Tasks

A real decision maker is

- *imperfect decision maker* , which is characterised (i) inability to specify all needed DM elements↰; (ii) perform in available time and with available computational resources all evaluations needed. Thus, tools are needed that (i) convert practically available knowledge ↰ and practically specified preferences↰ into DM elements↰; (ii) respect limited cognitive resources of decision maker.
- The subsequent sections use the approximation methodology (85), the minimum KLD principle↰ (90) and its generalised version (95) for constructing DM elements↰ from practically available knowledge pieces. They address the feature (i) of imperfect decision maker↰. The feature (ii) is addressed only fragmentally in Sections 25, 29.

Extension of Deterministic Models to Pds

- Prior non-probabilistic knowledge can often be expressed by restricting the simplex $\mathsf{f}^\star_\triangle$ (83) to the set $\mathsf{f}^\star$ in (88) via values $\mu_\kappa = 0$ of several functionals, indexed by $\kappa \in \kappa^\star = \{1, 2, \ldots, |\kappa^\star|\}, \ |\kappa^\star| < \infty$,

$$\mathsf{E}_\mathsf{f}[\phi_\kappa] = \int_{B^\star} \phi_\kappa(B)\mathsf{f}(B)\,\mathrm{d}B = 0. \tag{96}$$

- Indeed, decision makers often exploit deterministic models resulting from first principles and application-domain-specific knowledge. They are mostly expressed by a set of equations

$$\phi_\kappa(B) = \epsilon_\kappa(B), \tag{97}$$

where $\epsilon_\kappa(B)$, $\kappa \in \kappa^\star$, are modelling errors. The constraints (96) then simply express the expectation that modelling errors are unbiased.

- Known ranges $\epsilon^\star_\kappa(B)$ of modelling errors can be modelled in the same way. It suffices to take $\phi_\kappa(B) =$ indicator of the sets $\epsilon^\star_\kappa(B)$, $B \in B^\star$.

Extension of Deterministic Models to Pds (cont.)

- If the expectation that modelling errors are out of a given range is too high, the (second) moments serve well for error characterisation.
- Known ranges of the quantities forming the behaviour can be respected via range indicators or second moments, similarly as modelling errors.
- After specifying the set $\mathsf{f}^\star$, the minimum KLD principle (90) is applied and possibly followed by the approximation of the obtained $\mathsf{f} = {}^{O}\mathsf{f}$ by a feasible $\hat{\mathsf{f}}$ according to (85).
- The needed prior guess f_0 is mostly chosen as a soft delimitation of the support $B^\star$ of the involved pds.
- An algorithmic implementation may indeed support well an imperfect decision maker↰, e.g. [KBG+11].

Merging of Incompletely Compatible Pds: Problem

- The set $f^{\star}$, specified by conditions $E_f[\phi_\kappa] = 0,, \forall \kappa \in \kappa^{\star}$, can be empty when the processed knowledge pieces are incompatible. Then, a meaningful solution of (90) does not exist.
- By considering various "compatible" subsets of these conditions, say considering them individually, we get a collection of different pds $f_\kappa \in f^{\star}_{\triangle}$ (83) that have to be combined into a single representant $\hat{f}$.
- This is a prototype of the supporting merging DM task that has to be resolved when serving to an imperfect decision maker↰.
- Especially, the merging is believed to be an efficient tool for solving, otherwise extremely hard, problems of de-centralised decision making [BAHZ09]. The representant $\hat{f}$ of $(f_\kappa)_{\kappa\in\kappa^{\star}}$ is found via the generalised KLD principle (95).

Merging of Incompletely Compatible Pds: Formalisation

- The behaviour $B \in B^\star$ is assumed to be described by an unknown pd $\mathsf{f} \in \mathsf{f}^\star \subset \mathsf{f}^\star_\triangle$, i.e. the pair (B, F) forms ignorance↰ of the supporting merging DM task.
- The pd $\mathcal{A} = \mathcal{F}(\mathsf{f})$ is the action to be chosen.
- The knowledge↰ $\mathcal{K}_{\mathcal{F}^\star}$ is delimited by the specialisation of (93)

$$\begin{aligned} &\mathcal{K}_{\mathcal{F}^\star}: \\ &\mathsf{E}_{\mathcal{F}}[\mathsf{D}(\mathsf{f}_\kappa||\mathsf{f})] \leq \beta_\kappa < \infty,\ \kappa \in \kappa^\star = \{1, \ldots, |\kappa^\star|\},\ |\kappa^\star| < \infty, \\ &\mathcal{F}_0(\mathsf{f}) = \text{ prior (flat) guess of the action } \mathcal{F} \in \mathcal{F}^\star,\ \text{see (95)} \\ &\mathsf{f}_\kappa(B) \text{ are given pds in } \mathsf{f}^\star_\triangle,\ \text{see (83).} \end{aligned} \tag{98}$$

The constraints on the expected KLD of f_κ on f mean that pd f is an acceptable compromise with respect to a given f_κ, if the pd↰ f is a good approximation of the pd f_κ, cf. (85).

The generalised minimum KLD entropy principle guides the merging.

- Under constraints (98), the optimal action ${}^{O}\mathcal{F} \in \mathcal{F}^{\star}$ is defined by (95) and minimises the Kuhn-Tucker functional [KT51], given by multipliers $\lambda_{\kappa} \geq 0$,

$$
{}^{O}\mathcal{F} \in \mathrm{Arg} \min_{\mathcal{F} \in \mathcal{F}^{\star}} \int_{(B^{\star}, \mathsf{f}^{\star})} \mathcal{F}(\mathsf{f}) \tag{99}
$$

$$
\left[\mathsf{f}(B) \ln \left(\frac{\mathcal{F}(\mathsf{f})}{\mathcal{F}_0(\mathsf{f})} \right) + \sum_{\kappa \in \kappa^{\star}} \lambda_{\kappa} \mathsf{f}_{\kappa}(B) \ln \left(\frac{\mathsf{f}_{\kappa}(B)}{\mathsf{f}(B)} \right) \right] \mathrm{d}(B, \mathsf{f}).
$$

The minimiser providing the optimal solution has the form

$$
{}^{O}\mathcal{F}(\mathsf{f}) \propto \mathcal{F}_0(\mathsf{f}) \prod_{B \in B^{\star}} \mathsf{f}(B)^{\rho(B)} \text{ with}
$$

$$
\rho(B) = \sum_{\kappa \in \kappa^{\star}} \lambda_{\kappa} \mathsf{f}_{\kappa}(B), \ \lambda_{\kappa} \geq 0 \text{ respecting inequalities in (98)}
$$

and being zero when the bound is not reached.

- For the conjugate prior pd in the form of Dirichlet pd [Ber85, KBG+06],

$$\mathcal{F}_0(\mathsf{f}) \propto \prod_{B \in B^\star} \mathsf{f}(B)^{\nu_0(B)-1} \text{ with } \nu_0(B) > 0, \int_{B^\star} \nu_0(B)\,\mathrm{d}B < \infty,$$

the pd ${}^{O}\mathcal{F}(\mathsf{f})$ (93) is also Dirichlet pd given by $\nu(B) = \nu_0(B) + \rho(B) = \nu_0(B) + \sum_{\kappa \in \kappa^\star} \lambda_\kappa \mathsf{f}_\kappa(B)$.

- The Dirichlet pd determined by $\nu(B)$ has the expected value, which is a "point" representant of incompletely compatible pds f_κ, $\kappa \in \kappa^\star$,

$$\hat{\mathsf{f}}(B) = \mathsf{E}_{{}^{O}\mathcal{F}}[\mathsf{f}(B)] = \frac{\nu_0(B) + \sum_{\kappa \in \kappa^\star} \lambda_k \mathsf{f}_\kappa(B)}{\int_{B^\star} \nu_0(B)\,\mathrm{d}B + \sum_{\kappa \in \kappa^\star} \lambda_\kappa}. \qquad (100)$$

Is is convex combination of the merged pds and normalised $\nu_0(B)$.

Remarks on Merging

- The derived merging (100) justified and extends the heuristically motivated arithmetic pooling [Ber85, GKO05].
- A related derivation called supra-Bayesian merging can be found in [Seč10].
- The combination of pds (100) provides an invaluable tool for sharing knowledge and preferences among decision makers indexed by $\kappa \in \kappa^{\star}$ [KGBR09].
- Often, the decision makers can characterise the knowledge or preferences, possibly with help of minimum KLD principle, only via a conditional (marginal) version of the pd $\mathsf{f}_{\kappa}(B)$.
- *fragmental pd* is a common name we use for all possible marginal and conditional pds derived from $\mathsf{f}_{\kappa}(B)$.
- An extension of a fragmental pd to the pd on $B^{\star}$ is addressed in the subsequent text.

Extensions of Pds: Formalisation

- A finite collection of decision makers, indexed by $\kappa \in \kappa^\star$, operates on the behaviour↰ B that includes all quantities considered by them.
- A specific $\underline{\kappa}$th decision maker↰ splits the behaviour

$$B = (U_{\underline{\kappa}}, \mathcal{G}_{\underline{\kappa}}, \mathcal{K}_{\underline{\kappa}}) = \tag{101}$$

part (uninteresting for, modelled by, known to) $\underline{\kappa}$th decision maker.

- The pd ${}^{e}\mathsf{f}_{\underline{\kappa}}(B)$ extending the pd $\mathsf{f}_{\underline{\kappa}}(\mathcal{G}_{\underline{\kappa}}|\mathcal{K}_{\underline{\kappa}})$ is the action↰ $\mathcal{A}$ of this supporting extension DM task. It belongs to the set

$${}^{e}\mathsf{f}_{\underline{\kappa}} \in {}^{e}\mathsf{f}_{\underline{\kappa}}^{\star} \equiv \{\text{pds } {}^{e}\mathsf{f}(B) \text{ on } B^\star \text{ such that } {}^{e}\mathsf{f}(\mathcal{G}_{\underline{\kappa}}|\mathcal{K}_{\underline{\kappa}}) = \mathsf{f}_{\underline{\kappa}}(\mathcal{G}_{\underline{\kappa}}|\mathcal{K}_{\underline{\kappa}})\} \tag{102}$$

- The knowledge↰ $\mathcal{K}_{\mathcal{A}^\star}$ of the supporting DM task is the merger (100) evaluated for extensions of fragmental pd↰s

$${}^{e}\hat{\mathsf{f}}(B) = \frac{\nu_0(B) + \sum_{\kappa \in \kappa^\star} \lambda_k \, {}^{e}\mathsf{f}_{\kappa}(B)}{\int_{B^\star} \nu_0(B)\,\mathrm{d}B + \sum_{\kappa \in \kappa^\star} \lambda_\kappa}, \tag{103}$$

with $\lambda_\kappa > 0$ chosen so that $\mathsf{D}({}^{e}\mathsf{f}_\kappa || {}^{e}\hat{\mathsf{f}}) = \beta_\kappa$ or $\lambda_\kappa = 0$, $\kappa \in \kappa^\star$.

Extensions of Pds: Solution

The pd ${}^{e}\hat{f}(B)$ is the best representative (merger) of the supplied extensions. Thus, it makes sense to select $\underline{\kappa}$th extension as its best approximation.

Proposition 26 (The Optimal Extension)

The pd

$$ {}^{e}f_{\underline{\kappa}}(B) = {}^{e}\hat{f}(U_{\underline{\kappa}}|\mathcal{G}_{\underline{\kappa}}, \mathcal{K}_{\underline{\kappa}}) f_{\underline{\kappa}}(\mathcal{G}_{\underline{\kappa}}|\mathcal{K}_{\underline{\kappa}})\, {}^{e}\hat{f}(\mathcal{K}_{\underline{\kappa}}), \tag{104} $$

is the unique extension in the set (102) that minimises KLD↰ of ${}^{e}\hat{f}$ *on* ${}^{e}f_{\underline{\kappa}}$. *The pds* ${}^{e}\hat{f}(U_{\underline{\kappa}}|\mathcal{G}_{\underline{\kappa}}, \mathcal{K}_{\underline{\kappa}})$ *and* ${}^{e}\hat{f}(\mathcal{K}_{\underline{\kappa}})$ *are conditional and marginal pds of the pd* ${}^{e}\hat{f}(B)$.

Proof By a straightforward evaluation, see also [KGBR09]. □

Concluding Remarks

- The mapping of non-probabilistic knowledge or preferences on pds is a formally straightforward application of the minimum KLD principle possibly combined with approximation of the result by a feasible pd.
- Extension of the fragmental pds (104) and the merging (103) of several extensions lead to the implicit formula for the optimal merger ${}^{e}\hat{\mathsf{f}}(B)$

$$ {}^{e}\hat{\mathsf{f}}(B) = \frac{\nu_0(B) + \sum_{\kappa\in\kappa^\star} \lambda_k \, {}^{e}\hat{\mathsf{f}}(U_\kappa|\mathcal{G}_\kappa,\mathcal{K}_\kappa)\mathsf{f}_\kappa(\mathcal{G}_\kappa|\mathcal{K}_\kappa)\, {}^{e}\hat{\mathsf{f}}(\mathcal{K}_\kappa)}{\int_{B^\star}\nu_0(B)\,\mathrm{d}B + \sum_{\kappa\in\kappa^\star}\lambda_\kappa}, \quad (105) $$

  This implicit equation is conjectured to have, not necessarily unique, solution with unique fragmental pds ${}^{e}\hat{\mathsf{f}}(\mathcal{G}_\kappa|\mathcal{K}_\kappa)$, $\kappa \in \kappa^\star$.
- Algorithmic solution of the individual subtasks, especially of (105), is in its infancy.

Need for Approximations

- The presented methodology provides a complete formal solution of the optimal DM↰ under uncertainty. However, the solution of practically optimal design↰ is missing. The design complexity is not under a systematic control.
- This part introduces and discusses principles of that represent promising and well developed concepts of coping with the design complexity. Further on, just the direction called adaptive systems↰ is developed.

Core of Approximation Problems

- Parts 8 and 9 provide a quite general solution of DM uncertainty. The solution operates on the pd $\mathsf{f}_S(B) = \mathsf{M}(B)\mathsf{S}(B)$ (50) describing all possible realisations of the behaviour that includes all quantities considered by the decision maker within the time interval determined by the decision horizon.
- Within the considered FPD DM preferences are described by an ideal counterpart ${}^{\mathsf{I}}\mathsf{f}_S(B) = {}^{\mathsf{I}}\mathsf{M}(B)\,{}^{\mathsf{I}}\mathsf{S}(B)$ of the closed loop model $\mathsf{f}_S(B) = \mathsf{M}(B)\mathsf{S}(B)$.
- Thus, DM operates with a pair of scalar functions acting on the space $B^\star$ of an extremely high dimension, which makes the exact DM design exceptional.

Further text indicates the available directions for coping with the problem.

Distributed Systems: Solution Direction

Distributed DM systems are based on splitting an unmanageable DM task in DM problems dealing with subparts with respect to considered

- quantities,
- time horizon,
- domains of quantities,
- models considered,
- subproblems faced.

The key induced problem How to make splitting? has either heuristic or specific solutions for specific classes of problems.

Hierarchic Systems: Solution Direction

Distributed systems have to be equipped with a methodology that says How to "glue" together particular solutions?.
Practically, it is solved via hierarchic solutions in which "higher" DM levels influence lower levels using a sort of aggregation.
The key problems faced are the designs of

- the hierarchy structure,
- the aggregation ways allowing the upper levels to grasp practically DM elements.

Good solutions have to find a proper balance between achievable quality and scalability of the solution.
Note that inevitable uncertainties, additional delays, transaction costs connected with complex distributed and hierarchical dynamic systems may lead to unpredicted emerging behaviours. Simulations and techniques known from statistical physics are, for instance, used to such predictions. They serve well for analysis but not to design purposes. A general systematic design does not seem to exist.

Adaptive Systems: Solution Direction

- Adaptive systems use a specific feature of DM: the application of the strategies resulting from the design↰ requires them to be good at the realisation↰ of the behaviour↰, which start from the available realisation of the knowledge.
- Thus, it is sufficient to know a good approximation of the optimal strategy locally around the actual knowledge realisation. It can often be found. Such local approximations are known as adaptive systems [Kár98].
- Many features used by adaptive systems are of the distributed or/and hierarchical nature. Without commenting it, we stay with the adaptive systems↰ leaving the former directions out of the scope of this text.

vb_in_fpd

Adaptive Systems as Local Approximations

Remark 16

- *Note that there is no formal definition of adaptive systems. Their operational description is, for instance, in [AW89].*
- *We found the coined understanding of adaptive systems as local approximations very useful. It helps us to have a unified view on existing practical strategies and opens a way for designing novel ones [Kár98]. Moreover, it shows that the adaptive systems will be inevitably used in future due to the theoretically provable need for local approximations.*

Feasible and Approximate Learning

- The presented theory covers an extreme width of DM tasks. Just a few of them are solvable exactly and to some there are well established approximate solution techniques. Both are reviewed here.
- The summarised material does not cover full width of theory due to the limited
  - coverage by the contemporary research,
  - knowledge of lecturer,
  - time.
- The presentation focuses predominantly on learning based on parameter estimation with time invariant hidden quantities.

Factorised Parametric Model: Scalar-Observation Model Suffices

- The chain rule allows us to decompose any parametric model

$$\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}) = \prod_{i=1}^{\ell_\Delta} \mathsf{f}(\Delta_{t;i}|\Theta, A_t, \Delta_{t;i+1}, \ldots, \Delta_{t;\ell_\Delta}, \mathcal{K}_{t-1}). \quad (106)$$

- *factor* is the pd modelling a single entry $\Delta_{t;i}$ of observation in (106).
- A factor is the basic object we deal with further on as its use
  - is simpler than models predicting multivariate observations,
  - allows a fine modelling of individual observations (only a part of Θ can enter respective individual factors),
  - serves well for modelling of
  - *mixed observations* , which are vectors containing both continuous and discrete valued entries.

Simplified Notation of Learning Part

- In the factorised estimation, knowledge $\mathcal{K}_{t-1}$ is extended to $\mathcal{K}_{t;i} = \Delta_{t;i+1}, \ldots, \Delta_{t;\ell_\Delta}, \mathcal{K}_{t-1}$, where i points to the modelled observation entry.
- The pointer i and the additional condition $\Delta_{t;i+1}, \ldots, \Delta_{t;\ell_\Delta}$ are mostly dropped within this part. Formally, thus we deal with the parametric model $\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1})$ modelling a scalar observation Δ_t.
- Mostly, the parametric model (factor) is taken from the dynamic exponential family and mixtures of such models.

Dynamic Exponential Family: Definition

- *exponential family* (dynamic EF) consists of the parametric models, which can be written in the form

$$f(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}) = \mathsf{M}(\Psi_t, \Theta) = \mathsf{A}(\Theta) \exp \langle \mathsf{B}(\Psi_t), \mathsf{C}(\Theta) \rangle, \text{ given by} \tag{107}$$

- *data vector* $\Psi'_t \equiv [\Delta_t, \psi'_t]$ with $\ell_\Psi < \infty$; $'$ is transposition
- *regression vector* ψ_t, $\ell_\psi = \ell_\Psi - 1 < \infty$, whose values are in a known recursive way determined by the (enriched) knowledge $\mathcal{K}_{t-1}$ so that

$$(\Psi^\star_{t-1}, D^\star_t) = (\Psi^\star_{t-1}, A^\star_t, \Delta^\star_t) \to \Psi^\star_t \tag{108}$$

- $\langle \cdot, \cdot \rangle$ is the functional, linear in the first argument, typically,

$$\langle X, Y \rangle = \begin{cases} X'Y & \text{if } X,\ Y \text{ are vectors} \\ \mathrm{tr}[X'Y] & \text{if } X,\ Y \text{ are matrices, } \mathrm{tr} \text{ is trace} \\ \sum_{\iota \in \iota^\star} X_\iota Y_\iota & \text{if } X,\ Y \text{ are arrays with a multi-index } \iota, \end{cases} \tag{109}$$

- $\mathsf{A}(\cdot)$ is a nonnegative scalar function defined on $\Theta^\star$,
- $\mathsf{B}(\cdot)$, $\mathsf{C}(\cdot)$ are array functions of compatible, finite and fixed dimensions; they are defined on respective arguments in ${\Psi_t}^\star$ and $\Theta^\star$.

Remarks on EF

Remark 17

- *The definition of the exponential family↰ requires non-standardly the recursive updating of the data vector↰ Ψ_t. This recursion is the practically important condition for dynamic DM we deal with.*
- *Notice that equality is used in (107). The normalisation of this pd must not spoil the considered exponent form. This makes the allowed form rather restrictive. In the dynamic case with a nonempty regression vector ψ, ARX↰ (normal, linear-in-parameter) model and Markov chains↰ almost cover the exponential family.*

Textbooks Deal Mostly with Static EF

Some members of *static* EF characterised by empty regression vector are

Name	Parametric model	Observation Δ	Parameter
Exponential	$\frac{1}{\lambda}\exp\left(-\frac{\Delta}{\lambda}\right)$	$\in (0,\infty)$	$\lambda > 0$
Poisson	$\frac{\mu^{\Delta}}{\Gamma(\Delta+1)}\exp(-\mu)$	$\in \{0,1,\ldots\}$	$\mu > 0$
Multinomial	$\prod_{i\in\Delta^\star}\Theta_i^{\delta(i,\Delta)}$	$\in \{1,\ldots,\lvert\Delta^\star\rvert\}$	$\Big\{\Theta_\Delta \geq 0$
			$\sum_{\Delta\in\Delta^\star}\Theta_\Delta = 1\Big\}$
Normal	$\frac{1}{\sqrt{2\pi r}}\exp\left[-\frac{(\Delta-\mu)^2}{2r}\right]$	$\in (-\infty,\infty)$	$\mu \in (-\infty,\infty),\ r > 0$
Log-Normal	$\frac{1}{\Delta\sqrt{2\pi r}}\exp\left[-\frac{\ln^2\left(\frac{\Delta}{\mu}\right)}{2r}\right]$	$\in (0,\infty)$	$\mu > 0,\ r > 0$

- *Euler gamma function*
$$\Gamma(x) \equiv \int_0^\infty z^{x-1}\exp(-z)\,\mathrm{d}x < \infty \text{ for } x > 0. \tag{110}$$
- *Kronecker delta* is defined
$$\delta(i,\Delta) = \begin{cases} 1 & \text{if } i = \Delta \\ 0 & \text{if } i \neq \Delta \end{cases}. \tag{111}$$

The use of Proposition 15 reveals role of the exponential family.

Proposition 27 (Estimation and Prediction in Exponential Family)

Let natural conditions of DM↰ hold and the parametric model↰ belong to EF (107). Then, the predictive pd↰ has the form

$$
\begin{aligned}
f(\Delta_t|A_t, \mathcal{K}_{t-1}) &= \frac{J(V_{t-1} + B(\Psi_t), \nu_{t-1} + 1)}{J(V_{t-1}, \nu_{t-1})} && (112)\\
V_t = V_{t-1} + B(\Psi_t),\ V_0 = 0;&\quad \nu_t = \nu_{t-1} + 1,\ \nu_0 = 0 && (113)\\
J(V, \nu) &= \int_{\Theta^\star} A^{\nu}(\Theta) \exp \langle V, C(\Theta) \rangle\, f(\Theta)\, d\Theta, && (114)
\end{aligned}
$$

where $f(\Theta)$ *is a prior pd↰. The posterior pd↰ is*

$$
f(\Theta|\mathcal{K}_t) = \frac{A^{\nu_t}(\Theta) \exp \langle V_t, C(\Theta) \rangle\, f(\Theta)}{J(V_t, \nu_t)} \qquad (115)
$$

with the likelihood↰

$$
L(\Theta, \mathcal{K}_t) \equiv L(\Theta, V_t, \nu_t) = A^{\nu_t}(\Theta) \exp \langle V_t, C(\Theta) \rangle . \qquad (116)
$$

Finite-Dimensional Sufficient Statistic

The further discussion needs a few statistical terms:

- *statistic* is a (measurable) mapping acting on the estimation knowledge↰
  When there is no danger of misunderstanding, statistic and its values are not distinguished.
- *sufficient statistic* V meets the identity $\mathsf{f}(\Theta|\mathcal{K}_t) = \mathsf{f}(\Theta|\mathsf{V}(\mathcal{K}_t))$, i.e. instead of the knowledge realisation it suffices to store realisation of the statistic.
- *finite-dimensional statistic* maps estimation knowledge↰ into a finite-dimensional space whose dimension does not grow with increasing observation time t.

Remarks on EF

- Without the recursively updating the data vector↰ Ψ_t the estimation cannot be recursive, cf. (113).
- The posterior pd↰ for a parametric model↰ in the exponential family↰ has the fixed functional form (115), which is determined by the value of the finite-dimensional sufficient statistic↰ V_t, ν_t.
- EF essentially covers the set of parametric model↰s admitting a finite dimensional statistic↰. It is the only "smooth" class with the parametric-model support independent of Θ [Koo36]. The uniform parametric model has Θ-dependent support and admits finite-dimensional sufficient statistic, too. facestuniex
- The parameter estimation coincides with the data-updating part of filtering↰. It admits finite-dimensional sufficient statistic if the observation model↰ belongs to EF and the time evolution model↰ maps

$$\mathsf{f}(X_t|\mathcal{K}_t) \quad \propto \quad \mathsf{A}(X_t)^{\nu_{t|t+1}} \exp\left\langle V_{t|t+1}, \mathsf{C}(X_t)\right\rangle \tag{117}$$
$$\to \mathsf{f}(X_{t+1}|\mathcal{K}_t) \propto \mathsf{A}(X_{t+1})^{\nu_{t+1|t+1}} \exp\left\langle V_{t+1|t+1}, \mathsf{C}(X_{t+1})\right\rangle .$$

Such a class of models is inspected in [Dau88].

Estimation in EF with Conjugate Prior Pd

- *conjugate prior* pd $\mathsf{f}(\Theta)$ belongs to the same set $\mathsf{f}^\star$ of pds as the posterior pd.
  This definition makes a good practical sense if the set $\mathsf{f}^\star$ is (substantially) smaller than the set of all pds on $\Theta^\star$.
- The pd

$$\mathsf{f}(\Theta) \propto \mathsf{A}^{\nu_0}(\Theta) \exp \langle V_0, \mathsf{C}(\Theta) \rangle \chi_{\Theta^\star}(\Theta), \tag{118}$$

  determined by the finite-dimensional prior statistics V_0, ν_0 and a non-negative function $\chi_{\Theta^\star}(\Theta)$ is conjugate to the exponential family.
  With it, the prediction and estimation formulas (112) and (115) are valid if
  - V_0, ν_0 replace the zero initial conditions in (113),
  - the function $\chi_{\Theta^\star}(\cdot)$ is formally used as the prior pd.
- Mostly, $\chi_{\Theta^\star}(\cdot)$ is
- *set indicator* , which is equal to 1 on $\Theta^\star$ and zero otherwise.

ARX Model: Normal Autoregressive-Regressive Model Linear in Parameters with External Variables in Regression Vector

- *ARX model* describing ith factor is the parametric model ↰ described by the pd

$$
\begin{aligned}
& \mathsf{f}(\Delta_{t;i}|\Theta, A_t, \Delta_{t;i+1}, \ldots, \Delta_{t;\ell_\Delta}, \mathcal{K}_{t-1}) = \mathsf{M}(\Psi_t, \Theta) \\
& = \mathcal{N}_{\Delta_{t;i}}(\theta'\psi_t, r) = (2\pi r)^{-0.5} \exp[-0.5r^{-1}(\Delta_{t;i} - \theta'\psi_t)^2] \\
& = \underbrace{(2\pi r)^{-0.5}}_{\mathsf{A}(\Theta)} \exp\{\mathrm{tr}(\underbrace{\Psi_t\Psi_t'}_{\mathsf{B}(\Psi_t)} \underbrace{(-0.5[-1,\theta']'r^{-1}[-1,\theta'])}_{\mathsf{C}(\Theta)})\}
\end{aligned}
\tag{119}
$$

$$
\begin{aligned}
\Theta = (\theta, r) &= (\text{regression coefficient,noise variance}) \\
&= (\ell_\psi\text{-dimensional vector, positive scalar}) \\
\psi_t &= \psi(A_t, \Delta_{t;i+1}, \ldots, \Delta_{t;\ell_\Delta}, \mathcal{K}_{t-1}) = \text{regression vector} \\
\Psi_t &= [\Delta_{t;i}, \psi_t']' = \text{data vector} ↰
\end{aligned}
$$

Justification of ARX Model

- *innovations* $\varepsilon_t = \Delta_t - \mathsf{E}[\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}]$ form zero-mean sequence, uncorrelated with quantities in the conditioning.
- ARX model is obtained by
  - assuming a finite constant conditional variance r of innovations.
  - assuming negligible errors $\mathsf{E}[\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}] \approx \theta'\psi_t$ (Taylor expansion)
  - selecting the pd of innovations according to the maximum entropy principle↰.
  - bijectively transforming $\varepsilon_t \leftrightarrow \Delta_t$
- Regression vector ψ_t of ith factor is any known non-linear function of the action and of the (enriched) knowledge $A_t, \Delta_{t;i+1}, \ldots, \Delta_{t;\ell_\Delta}, \mathcal{K}_{t-1}$ that allows a recursive evaluation of the data vector↰.

Parameter Estimation for ARX Model

- The likelihood (115) becomes $\mathsf{L}(\Theta, \mathcal{K}_t) \equiv \mathsf{L}(\theta, r, V_t, \nu_t)$

$$= \quad (2\pi r)^{-0.5\nu_t} \exp(-0.5 r^{-1}[-1, \theta'] V_t [-1, \theta']') \qquad (120)$$

having as the conjugate prior

- *GiW* Gauss-inverse-Wishart (Gauss-inverse-Gamma) pd

$$\mathcal{G}i\mathcal{W}(V_0, \nu_0) \equiv \frac{(2\pi r)^{-0.5(\nu_0+\ell_\psi+2)} \exp(-0.5 r^{-1}[-1, \theta'] V_0 [-1, \theta']')}{\mathsf{J}(V_0, \nu_0)} \qquad (121)$$

$$\mathsf{J}(V, \nu) \quad = \quad \Gamma(0.5\nu)\left({}^{\Delta}V - {}^{\psi\Delta}V' \, {}^{\psi}V^{-1} \, {}^{\psi\Delta}V\right)^{-0.5\nu} \left| {}^{\psi}V \right|^{-0.5} 2^{0.5\nu} (2\pi)^{0.5\ell_\psi}$$

$$V \quad = \quad \begin{bmatrix} {}^{\Delta}V & {}^{\psi\Delta}V' \\ {}^{\psi\Delta}V & {}^{\psi}V \end{bmatrix}, \qquad (122)$$

is finite for a positive definite V ($V > 0$) and positive $\nu > 0$.

- Estimation provides posterior pds preserving GiW form $\mathcal{G}i\mathcal{W}(V_t, \nu_t)$

$$V_t = V_{t-1} + \Psi_t \Psi_t' > 0, \;\; \nu_t = \nu_{t-1} + 1 > 0 \qquad (123)$$

initiated by the statistic values of the conjugate prior (121).

Relation to Least Squares (LS)

Let x denote $2r$ multiple of the exponent of the likelihood (120). It describes the posterior pd obtained for the flat prior with $V_0 = 0$, $\nu_0 = 0$.

$$x = \sum_{\tau=1}^{t} \underbrace{(\Delta_\tau - \theta' \psi_t)^2}_{prediction\ \ error} = \Lambda_t + (\theta - \hat{\theta}_t)' C_t^{-1} (\theta - \hat{\theta}_t) \tag{124}$$

$$C_t = \left(\sum_{\tau=1}^{t} \psi_\tau \psi_\tau' \right)^{-1} = \text{LS covariance}$$

$$\hat{\theta}_t = C_t \sum_{\tau=1}^{t} \psi_\tau \Delta_\tau = C_t\, {}^{\psi\Delta}V_t = \text{LS parameter estimate}$$

$$\Lambda_t = {}^{\Delta}V_t - {}^{\psi\Delta}V_t'\, {}^{\psi}V_t^{-1}\, {}^{\psi\Delta}V_t = {}^{\Delta}V_t - \hat{\Theta}_t' C_t^{-1} \hat{\Theta}_t' = \text{LS remainder}$$

$$\hat{\theta}_t = \mathsf{E}[\theta | V_t, \nu_t], \text{ coincidence is valid for } V_0 = 0, \nu_0 = 0$$

$$\hat{r}_t = \mathsf{E}[r | V_t, \nu_t] = \frac{\Lambda_t}{\nu_t - 2}, \text{ coincidence is valid for } V_0 = 0, \nu_0 = 0$$

$$\hat{r}_t C_t = \mathrm{cov}[\Theta | V_t, \nu_t], \text{ coincidence is valid for } V_0 = 0, \nu_0 = 0.$$

Comments on Parameter Estimation for ARX Model

- *extended information matrix* is the name used for the statistics V_t. The recursion for this matrix $V_t = V_{t-1} + \Psi_t \Psi_t'$ can be algebraically transformed into
- *RLS* , recursive least squares, update $\hat{\theta}_t$, C_t, $\hat{r}_t$, [Pet81].
- Usually, $\hat{\theta}_t$ and $\hat{r}_t$ are interpreted as the best point estimates of θ and r. For us, they form a part of sufficient statistic↰.
- The relation of RLS to the posterior pd and general asymptotic of learning, Proposition 18, provide rich asymptotic results for RLS.
- The non-trivial prior pd given by $V_0 > 0, \nu_0 > 0$ guarantees that $V_t > 0, \nu_t > 0$: the prior pd↰ regularises the posterior pd↰. In spite of this RLS are numerically sensitive and problem is addressed by using
- *LDL' decomposition* of extended information matrix↰ $V = LDL'$, L lower triangular matrix with unit diagonal, D diagonal matrix with positive diagonal, [Bie77, Pet81, GV89, KBG+06].

- If the data vector $\Psi_t \in \Psi_t{}^\star = (\Delta^\star, \psi^\star)$ of a factor has a finite amount of realisations $|\Psi^\star| < \infty$ then it is modelled by
- *Markov chain* , which is the parametric model described by the pd

$$f(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}) = \prod_{\Psi\in\Psi^\star} \Theta_{\Delta|\psi}^{\delta(\Psi-\Psi_t)} = \exp \sum_{\Psi\in\Psi^\star} \underbrace{\delta(\Psi-\Psi_t)}_{B_{\Delta|\psi}(\Psi)} \underbrace{\ln(\Theta_{\Delta|\psi})}_{C_{\Delta|\psi}(\Theta)}$$

$$\Theta \in \Theta^\star = \left\{\Theta_{\Delta|\psi} > 0, \sum_{\Delta\in\Delta^\star} \Theta_{\Delta|\psi} = 1 \ \forall\psi \in \psi^\star\right\} \quad (125)$$

The conjugate prior pd is

- *Dirichlet pd* is defined on $\Theta^\star$ (125)

$$f(\Theta) = \mathcal{D}i_\Theta(V_0) = \prod_{\psi\in\psi^\star} \frac{\prod_{\Delta\in\Delta^\star} \Theta_{\Delta|\psi}^{V_{0;\Delta|\psi}-1}}{\mathrm{Be}(V_{0;\cdot|\psi})} \quad (126)$$

$$\mathrm{Be}(V_{\cdot|\psi}) = \frac{\prod_{\Delta\in\Delta^\star} \Gamma(V_{\Delta|\psi})}{\Gamma(\sum_{\Delta\in\Delta^\star} V_{\Delta|\psi})}, \quad \Gamma(x) = \int_0^\infty z^{x-1}\exp(-z)\,\mathrm{d}z, \ x > 0.$$

For the Markov Chain, the posterior pd is Dirichlet pd $\mathcal{D}i_{\Theta}(V)$ given by

- *occurrence matrix* $V = (V_{\Delta|\psi} > 0)_{\Psi \in \Psi^\star}$ updates as follows

$$V_{t;\Delta|\psi} = V_{t;\Delta|\psi} + \delta(\Psi - \Psi_t). \tag{127}$$

- The corresponding predictive pd [KBG[+]06] has the form

$$f(\Delta|A, \psi, \mathcal{K}_{t-1}) = \mathsf{E}[\Theta_{\Delta|\psi}|A, \psi, V_{t-1}] = \frac{V_{t-1;\Delta|\psi}}{\sum_{\Delta \in \Delta^\star} V_{t-1;\Delta|\psi}} \tag{128}$$

$= \;$ relative frequency of occurrence of the data vector $\Psi = [\Delta, \psi']'$.

- The formula (128) relates the "classical" (frequency based) view on probabilities to the presented Bayesian theory. For instance, the asymptotic result on learning, Proposition 18, describes conditions under which the relative frequencies converge to unknown probabilities.

Estimation out of EF

The EF↰ and special uniform pds provide a basic supply of dynamic factors admitting finite-dimensional sufficient statistic↰. What can be done for other parametric models?
Under natural conditions of DM↰ (45), generalised Bayesian estimation↰, Proposition 15, updates the posterior pds according to the Bayes rule↰ (54)

$$\mathsf{f}(\Theta|\mathcal{K}_t) = \frac{\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1})\mathsf{f}(\Theta|\mathcal{K}_{t-1})}{\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1})}, \ t \in t^\star.$$

Out of EF↰ (107), the complexity of these pds increases quickly with increasing t.
This section inspects the recursive estimation applicable out of EF. The outlined equivalence approach [Kul93, Kul94, Kul96] addresses the problem systematically.

Recursively Feasible Representation of Pds

- Considered cases do no admit sufficient statistic↰, thus instead of $f(\Theta|\mathcal{K}_{t-1})$, we have to deal with its approximation by a pd $\hat{f}(\Theta|V_{t-1})$ of a fixed functional form and determined by a finite dimensional statistic V_{t-1}.
- A given pd $\hat{f}(\Theta|V_{t-1})$ can be seen as approximation of a whole set $f^{\star}(\Theta|\mathcal{K}_{t-1})$ of possible posterior pds.
- First, we search for $\hat{f}(\Theta|V_{t-1})$ that can be updated recursively and includes the exact posterior pd in the discussed equivalence set.

Bayes-Rule Compatible Statistic

Proposition 28 (Equivalence-Preserving Mapping)

Let $\mathsf{f}^\star(\Theta|\mathcal{K}_{t-1})$ *be a set of pds* $\mathsf{f}(\Theta|\mathcal{K}_{t-1})$ *with a common, time, data, and parameter invariant support↰. Let the mapping*

$$\mathsf{V}_t : \mathsf{f}^\star(\Theta|\mathcal{K}_{t-1}) \to V^\star_{t-1} \tag{129}$$

assign to each pd $\mathsf{f}(\Theta|\mathcal{K}_{t-1})$ *from* $\mathsf{f}^\star(\Theta|\mathcal{K}_{t-1})$ *a finite-dimensional statistic* $V_{t-1} \equiv \mathsf{V}_t(\mathcal{K}_{t-1})$ *"representing" it. Then, the value of* V_{t-1} *can be exactly recursively updated using only its previous value and the current parametric model↰* $\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1})$ *iff* V_t *is a time-invariant linear mapping* $\mathsf{V}_t \equiv \mathsf{V}$, $t \in t^\star$, *acting on logarithms of the pds involved. The logarithmic pds are treated as functions of* Θ.
V *has to map* Θ*-independent elements to zero.*

- *Riezs representation of* V, *[Rao87b], is – with an abuse of notation –* $\mathsf{V}(\ln(\mathsf{f}(\Theta|\mathcal{K}_{t-1}))) = \int_{\Theta^\star} \mathsf{V}(\Theta) \ln(\mathsf{f}(\Theta|\mathcal{K}_{t-1}))\, \mathrm{d}\Theta$, $\int_{\Theta^\star} \mathsf{V}(\Theta)\, \mathrm{d}\Theta = 0$.

Proof of the Sufficiency

Proof To demonstrate necessity is rather difficult, and the interested reader is referred to [Kul90a, Kul90b]. To show sufficiency of conditions on $\mathsf{V}_t \equiv \mathsf{V}$, $t \in t^\star$ it suffices to apply such V on the logarithmic version of the Bayes rule (54) and use both time invariance and linearity of V. The normalising term $\ln(\mathsf{f}(\Delta_t|A_t, \mathcal{K}_{t-1}))$ is independent of Θ and as such mapped to zero. The recursion for values of V_t is then

$$\begin{aligned} V_t &= \mathsf{V}\left[\ln\left(\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1})\right)\right] + V_{t-1}, \quad \text{with} \qquad (130) \\ V_0 &= \mathsf{V}[\ln(\mathsf{f}(\Theta))] \equiv \mathsf{V}(\ln(\text{prior pd}\,\text{↰})). \end{aligned}$$

□

Formula (130) is the true recursion if we need not store complete past observed data for evaluating the parametric model↰ $\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1})$. Thus, as for EF↰, we consider models $\mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1}) = \mathsf{M}(\Psi_t, \Theta)$ depending on a recursively updatable data vector↰ Ψ_t.

Approximation in Data-Vector Space

- The unknown posterior pd should be approximated using the (generalised) minimum KLD principle. Instead of this non-elaborated way, the problem is transformed into approximation of the unknown
- *empirical pd of data vector*

$$\mathsf{f}_t(\Psi) \equiv \frac{1}{t}\sum_{\tau=1}^{t}\delta(\Psi - \Psi_\tau),\ \Psi \in \Psi^\star \equiv \bigcup_{t\in t^\star} \Psi_t{}^\star. \qquad (131)$$

- The value of statistic V_t (130) has the alternative expression

$$V_t = t\int_{\Psi^\star} \mathsf{f}_t(\Psi) \underbrace{\int_{\Theta^\star} \mathsf{V}(\Theta)\ln(\mathsf{M}(\Psi,\Theta))\,\mathrm{d}\Theta}_{\mathsf{h}(\Psi)}\,\mathrm{d}\Psi + V_0. \qquad (132)$$

- This value can be evaluated recursively

$$V_t = V_{t-1} + \mathsf{h}(\Psi_t). \qquad (133)$$

Recursively Feasible Approximation of Empirical Pd

- The posterior pd can be given the form

$$\mathsf{f}(\Theta|\mathcal{K}_t) \propto \mathsf{f}_0(\Theta)\exp\left[t\int_{\Psi^\star}\mathsf{f}_t(\Psi)\ln(\mathsf{M}(\Psi,\Theta))\,\mathrm{d}\Psi\right]. \tag{134}$$

- We search for the approximate posterior pd in the form

$$\hat{\mathsf{f}}(\Theta|\mathcal{K}_t) \propto \mathsf{f}_0(\Theta)\exp\left[t\int_{\Psi^\star}\hat{\mathsf{f}}_t(\Psi)\ln(\mathsf{M}(\Psi,\Theta))\,\mathrm{d}\Psi\right]. \tag{135}$$

- The estimate $\hat{\mathsf{f}}_t(\Psi)$ of the unknown empirical pd of the data vector $\mathsf{f}_t(\Psi)$ minimising the KLD $\mathsf{D}(\hat{\mathsf{f}}||\hat{\mathsf{f}}_0)$ under the informational constraint

$$\int_{\Psi^\star}\hat{\mathsf{f}}(\Psi)\mathsf{h}(\Psi)\,\mathrm{d}\Psi = (V_t - V_0)/t \tag{136}$$

has the form

$$\hat{\mathsf{f}}_t(\Psi) \propto \hat{\mathsf{f}}_0(\Psi)\exp[\lambda_t'\mathsf{h}(\Psi)], \tag{137}$$

where the multipliers λ_t are chosen so that (136) is met for $\hat{\mathsf{f}} = \hat{\mathsf{f}}_t$.

Algorithmic Summary

- Off line phase consists of selecting
  - parametric model↰ $\mathsf{f}(\Delta|\Theta, A_t, \mathcal{K}_{t-1}) = \mathsf{M}(\Psi_t, \Theta)$ with data vector↰ Ψ_t,
  - kernel $\mathsf{V}(\Theta)$ defining Riezs representation↰
  - an algorithm evaluating functions $\mathsf{h}(\Psi) = \int_{\Theta^\star} \mathsf{V}(\Theta) \ln(\mathsf{M}(\Psi, \Theta))\, \mathrm{d}\Theta$,
  - a prior pd↰ $\mathsf{f}(\Theta)$ defining $V_0 = \int_{\Theta^\star} \mathsf{V}(\Theta) \ln(\mathsf{f}(\Theta))\, \mathrm{d}\Theta$,
  - a prior (flat) guess $\hat{\mathsf{f}}_0(\Psi)$ of the empirical pd of data vector↰,
- On line phase runs for $t \in t^\star$ when Ψ_t are recursively updated
  - the stored statistic is updated $V_t = V_{t-1} + \mathsf{h}(\Psi_t)$,
  - the empirical pd is approximated by $\hat{\mathsf{f}}_t(\Psi) \propto \hat{\mathsf{f}}_0(\Psi) \exp[\lambda_t' \mathsf{h}(\Psi)]$, where the multipliers λ_t are chosen so that (136) is met.
  - The posterior pd is approximated by (135)
    $\hat{\mathsf{f}}(\Theta|\mathcal{K}_t) \propto \mathsf{f}_0(\Theta) \exp\left[t \int_{\Psi^\star} \hat{\mathsf{f}}_t(\Psi) \ln(\mathsf{M}(\Psi, \Theta))\, \mathrm{d}\Psi\right]$.

Remarks I

- The kernel V, which can be a vector generalised function [Vla79], represents the key tuning knob of the approach. Options leading to discrete versions of the function and/or its derivatives, or $\mathrm{M}(\Psi_i, \Theta)$ on a grid of Ψ_i have been tried with a success, but a deeper insight is needed.
- The name "equivalence approach" stresses the fact that the set of posterior pds $\mathrm{f}^\star$ splits to equivalence classes. The posterior pds with the same V_t cannot be distinguished.
- The required commutation of the mapping V with the data updating of the posterior pds is crucial. The recursion for V_ts is exact and the approximation errors caused by the use of $\hat{\mathrm{f}}(\Theta|V_t)$ instead of $\mathrm{f}(\Theta|\mathcal{K}_t)$ do not accumulate! Use of a noncommutative projection $\mathrm{V}_t : \mathrm{f}^\star(\Theta|\mathcal{K}_t) \to V_t^\star$ is always endangered by a divergence as the estimation described by the Bayes rule can be viewed as a dynamic system evolving the functions $\ln(\mathrm{f}(\Theta|\mathcal{K}_{t-1}))$ at the stability boundary.

Remarks II

- The algorithm defining the vector function $\mathsf{h}(\Psi)$ via integrations represent the computationally most demanding part of the algorithm. The integrations can be performed in off-line mode if their results can be efficiently stored (the resulting functions interpolated).
- The solution of the nonlinear equation for Lagrangian multipliers λ_t is also hard, but it is a standard problem.
- We would like to get the exact posterior pd if the model belongs to the exponential family (107). This dictates the choice of the mapping V that should make $\mathsf{h}(\Psi)$ a bijective image of $[\mathsf{B}(\Psi), 1]$. It is sufficient, to introduce the prior initial moments of the vector function $\mathsf{V}(\Theta) \equiv [\mathsf{C}(\Theta), \ln(\mathsf{A}(\Theta))]$.

Tracking of Slowly Varying Parameters

- The parameter estimation relies on time-invariance of parameters. If this assumption is violated, the Bayesian filtering↰ is to be used. It requires time evolution model↰ and its exact feasibility is even more restricted than the parameter estimation. This stimulated interest in intermediate case between parameter estimation and filtering, in
- *parameter tracking* , which is estimation of slowly varying parameters $\Theta_t \approx \Theta_{t-1}$ with a simplified specification of time evolution model↰.
- Parameter tracking forms the core of many adaptive systems↰. It modifies local model according to realisation↰s of behaviour.
- Parameter tracking, approximate evaluation of pds $\mathsf{f}(\Theta_t|\mathcal{K}_t)$, is based on a group of techniques called
- *forgetting* , which tries to exploit for estimation of Θ_t the valid part of $\mathcal{K}_t$ and discard invalid, typically obsolete, knowledge, [Pet81, KK84, Kul86, Kul87, KZ93, MK95, KK96, CS00, KA09].
- Here, we present the most advanced technique based on the developed approximation, Section 20, and Bayesian testing of hypotheses.

The relevant DM elements ↰ are

- the behaviour ↰
  $$\mathcal{B} = \big(\underbrace{(X^h, (\mathsf{f}(\Theta_t|\mathcal{K}_{t-1}))_{t\in t^\star})}_{hiddens}, \underbrace{(\hat{\mathsf{f}}(\Theta_t|\mathcal{K}_{t-1}))_{t\in t^\star}}_{action}, \underbrace{D^h, \mathcal{K}_0}_{knowledge}\big) =$$
  ((time-varying parameters,exact posterior pds), optional approximating pds, data records, prior knowledge),
- the aim is to evaluate recursively $\hat{\mathsf{f}}(\Theta_t|\mathcal{K}_{t-1}) \approx \mathsf{f}(\Theta_t|\mathcal{K}_{t-1})$ for $t \in t^\star$,
- observation model ↰ is a given pd $\mathsf{f}(\Delta_t|\Theta_t, A_t, \mathcal{K}_{t-1})$,
- time evolution model ↰ is unspecified but $\mathsf{f}(\Theta_{t+1} = \Theta|\mathcal{K}_t)$ is hoped to be close to the data-updated approximation
  $$\tilde{\mathsf{f}}(\Theta_{t+1} = \Theta|\mathcal{K}_t) \propto \mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1})\hat{\mathsf{f}}(\Theta_t = \Theta|\mathcal{K}_{t-1}) \Rightarrow \quad (138)$$
  $$\mathsf{D}(\mathsf{f}(\Theta_{t+1}|\mathcal{K}_t)||\tilde{\mathsf{f}}(\Theta_{t+1}|\mathcal{K}_t)) \leq \gamma_{t+1} < \infty. \quad (139)$$
- prior pd ↰ $\mathsf{f}(\Theta_1|\mathcal{K}_0)$ describing prior knowledge ↰ about Θ_1 is given,
- prior knowledge includes also the externally supplied (flat) alternative pd $({}^{\mathsf{a}}\mathsf{f}(\Theta_{t+1}|\mathcal{K}_t))_{t\in t^\star}$ and values γ_{t+1}.

Solution

- The minimum KLD principle, Section 21, with the alternative pd ${}^{a}\mathsf{f}(\Theta_{t+1}|\mathcal{K}_t)$ playing the role of a (flat) prior guess, and the inequality (139) constraint, delimiting the knowledge passed from previous time step, provide the optimal solution

$$\hat{\mathsf{f}}(\Theta_{t+1}|\mathcal{K}_t) \propto \tilde{\mathsf{f}}(\Theta_{t+1}|\mathcal{K}_t)^{\lambda_t}\, {}^{a}\mathsf{f}(\Theta_{t+1}|\mathcal{K}_t)^{1-\lambda_t}. \tag{140}$$

  The range $\lambda_t \in [0,1]$,, that depends on the bound (139), results from the addressed simple optimisation with a non-negative Kuhn-Tucker multiplier.

Remarks on Forgetting I

- λ is called forgetting factor. It controls compromise between the posterior pd obtained under the hypothesis that Θ_t is time invariant and an externally supplied alternative pd ${}^{a}\mathsf{f}$. The closer λ is to unity, the slower changes are expected, i.e. the higher weight the posterior pd corresponding to the time invariant case gets.
- The older are data built in through the parametric model the stronger flattening is applied to its values. Consequently, the older data influence the estimation results less than the new ones. Data are gradually "forgotten".
- For ${}^{a}\mathsf{f} \propto 1$ and $\lambda < 1$, the time evolution reduces to flattening of the pd obtained after data updating. It is intuitively appealing as our uncertainty about the estimated parameters can hardly decrease without knowing a good time evolution model (44) and with no new information processed.

Remarks on Forgetting II

- The alternative pd ${}^{a}\mathsf{f}$, typically, delimits where Θ_t can shift. The prior pd is a typical, reasonably conservative, choice of the alternative pd. Such nontrivial alternative pd prevents us to forget the "guaranteed" information. This stabilises whole learning and reflects very positively in its numerical implementations. Without this, the posterior pd may become too flat whenever the information brought by new data is poor.
- Note that lack of information brought by new data is more rule than exception. It is true especially for regulation [Mos94] that tries to make the closed control loop as quiet as possible: it tries to suppress any new information brought by data.
- In the extreme case of uniform alternative, the solution is called exponential forgetting otherwise it is called stabilised forgetting.

Remarks on Forgetting III

- The considered data updating of $\hat{f}$ (138) models slow variations. This model can be enriched by assuming at least partial variations. For instance, time invariance is admitted with some probability only and the description by the alternative pd is considered otherwise.
- The forgetting operation (140) preserves the basic property of the time updating: the posterior pd on parameters propagates without obtaining any new measured information.
- The forgetting factor λ can be either taken as a tuning knob or estimated. The predictive pd parameterised by it, however, depends on it in a very complex way so that a partitioned estimation has to be applied when its posterior pd is estimated on a pre-specified grid [ME76].
- The practical importance of this particular case of estimating slowly varying parameters cannot be over-stressed: the vast majority of adaptive systems rely on a version of forgetting.

Forgetting in EF

- Let the parametric model belongs to EF and the conjugate pd is considered given by sufficient statistic (V_t, ν_t).
- Let us allow slow parameter changes with the forgetting factor $\lambda \in [0,1]$ and the alternative conjugate pd given by the sufficient statistic ${}^{a}V_t$, ${}^{a}\nu_t$. Then, the prediction and estimation formulas, Proposition 27, remain unchanged with statistics evolving according to the recursion

$$\begin{aligned} V_t &= \lambda(V_{t-1} + \mathrm{B}(\Psi_t)) + (1-\lambda)\,{}^{a}V_t, \;\; V_0 \text{ given}, \\ \nu_t &= \lambda(\nu_{t-1} + 1) + (1-\lambda)\,{}^{a}\nu_t, \;\; \nu_0 \text{ given}. \end{aligned}$$

… Omitted Topics

- Useful examples like Kalman filtering [Pet81], i.e. stochastic filtering with linear Gaussian models. Other classes include linear models with restricted support [Pav08] and finite mixtures with factor↰s from exponential family↰ [KBG$^+$06].
- Monte Carlo techniques and their sequential variants known as particle filters [BS04].
- So called variational Bayes approximating a joint pds by product of conditionally independent factors [ŠQ05].

Feasible and Approximate Design

- The presented theory covers formally extreme width of tasks. Just a few of them are solvable exactly and to some there well established approximate solution techniques. Both are reviewed here.
- The summarised material does not cover full width of theory due to limited
  - coverage by the contemporary research,
  - knowledge of lecturer,
  - time.
- The presentation focuses predominantly on data-based design ↰ combined with parameter estimation, i.e. time invariant hidden quantities.

Evaluation Problem

- The optimal data-driven FPD↰ is described by Proposition 20, which provides the optimal randomised decision rule↰s

$$
\begin{aligned}
{}^{O}\mathsf{f}(A_t|D^{t-1}) &= {}^{I}\mathsf{f}(A_t|D^{t-1})\frac{\exp[-\omega_\gamma(A_t, D^{t-1})]}{\gamma(D^{t-1})}, \ \gamma(D^h) = 1 \\
\gamma(D^{t-1}) &\equiv \int_{A_t^\star} {}^{I}\mathsf{f}(A_t|D^{t-1}) \exp[-\omega(A_t, D^{t-1})] \, \mathrm{d}A_t, \quad \text{if } t < h \\
\omega_\gamma(A_t, D^{t-1}) &\equiv \int_{\Delta_t^\star} \mathsf{f}(\Delta_t|A_t, D^{t-1}) \ln \left(\frac{\mathsf{f}(\Delta_t|A_t, D^{t-1})}{\gamma(D^t)\, {}^{I}\mathsf{f}(\Delta_t|A_t, D^{t-1})} \right) d\Delta_t.
\end{aligned}
$$

- While the evaluation of high-dimensional integrals can be conceptually solved via Monte-Carlo techniques, the storing high dimensional functions like $\omega(A_t, D^{t-1})$ is known to be computationally hard.
- Practical, possibly approximate, evaluation of this strategy is discussed starting from analytically feasible cases followed by common approximation tricks adopted.

Finite-Dimensional Information State

- The predictive pd ↰ is obtain via parameter estimation ↰, Section 25. The feasible procedures led to $\mathsf{f}(\Delta_t|A_t,\mathcal{K}_{t-1}) \approx \mathsf{f}(\Delta_t|\psi_t, V_{t-1})$, where the regression vector ↰ ψ and the value of the sufficient statistic ↰ V_{t-1} are finite-dimensional and allow recursive updating. They form
- *information state* $X_{t-1} = (\Psi_{t-1}, V_{t-1})$, which can be recursively updated and knowledge ↰ $\mathcal{K}_{t-1} = X_{t-1} = (\Psi_{t-1}, V_{t-1})$ is finite-dimensional. The predictive pd ↰ and rule $X^\star_{t-1} \rightarrow_{D_t} X^\star_t$ define state model $\mathsf{f}(X_t|A_t, X_{t-1})$ with an observable information state.
- Without loss of generality the ideal pd ↰ can be chosen so that it depends on the information state, too. Consequently,
    - the functions occurring in FPD depend on the finite-dimensional information state: $\gamma(D^{t-1}) = \gamma(X_{t-1})$, $\omega(A_t, D^{t-1}) = \omega(A_t, X_{t-1})$.
    - the optimal strategy explicitly influences both the primary quantities to be intentionally influenced and learning process: this property was observed in [Fel60, Fel61] and called dual control.

FPD with Finite Number of Behaviour Realisations

- If the information state X_t and actions A_t have finite numbers of realisations then the Proposition 20 provides directly the optimum

$$
\begin{aligned}
{}^{O}\mathsf{f}(A_t|X_{t-1}) &= {}^{I}\mathsf{f}(A_t|X_{t-1})\frac{\exp[-\omega_\gamma(A_t,X_{t-1})]}{\gamma(X_{t-1})},\ \gamma(X_h)=1 \qquad (141)\\
\gamma(X_{t-1}) &\equiv \sum_{A\in A_t^\star} {}^{I}\mathsf{f}(A_t|X_{t-1})\exp[-\omega(A_t,X_{t-1})],\quad \text{if } t<h\\
\omega_\gamma(A_t,X_{t-1}) &\equiv \sum_{X\in X_t^\star} \mathsf{f}(X_t|A_t,X_{t-1})\ln\left(\frac{\mathsf{f}(X_t|A_t,X_{t-1})}{\gamma(X_t)\,{}^{I}\mathsf{f}(X_t|A_t,X_{t-1})}\right).
\end{aligned}
$$

- All functions $\omega(A_t,X_{t-1})$, $\gamma(X_t)$ etc. are tables with amount of entries given by $|A^\star|$ and $|X^\star|$. The complexity of evaluations (summing over $A^\star$, $X^\star$, storing of the tables) is also implied by them.
- FPD is simple for small $|A^\star|$, $|X^\star|$ and infeasible for large $|A^\star|$, $|X^\star|$.
- The evaluation is slightly simplified if the stationary strategy, obtained for horizon $h\to\infty$, Proposition 13, is considered.

FPD with Linear Gaussian Information State: Formulation

- Let the (observable information) state X_t be described by the linear Gaussian (LG) model

$$\begin{aligned} \mathsf{f}(X_t|A_t, X_{t-1}) &= \mathcal{N}_{X_t}(\mathbf{A}X_{t-1} + \mathbf{B}A_t, {}^{X}\mathbf{R}) \\ \mathcal{N}_X(\mathbf{M}, \mathbf{R}) &= |2\pi\mathbf{R}|^{-0.5}\exp[-0.5(X-\mathbf{M})'\mathbf{R}^{-1}(X-\mathbf{M})] \end{aligned} \tag{142}$$

  determined by known matrices $(\mathbf{A}, \mathbf{B}, {}^{X}\mathbf{R})$.

- Let the ideal pd be also the linear Gaussian (LG) model

$$\begin{aligned} {}^{\mathsf{I}}\mathsf{f}(X_t|A_t, X_{t-1}) &= \mathcal{N}_{X_t}({}^{\mathsf{I}}\mathbf{A}X_{t-1} + {}^{\mathsf{I}}\mathbf{B}A_t, {}^{\mathsf{I}}\mathbf{R}) \\ {}^{\mathsf{I}}\mathsf{f}(A_t|X_{t-1}) &= \mathcal{N}_{A_t}({}^{\mathsf{I}}\mathbf{C}X_{t-1}, {}^{A\mathsf{I}}\mathbf{R}) \end{aligned} \tag{143}$$

  determined by known matrices $({}^{\mathsf{I}}\mathbf{A}, {}^{\mathsf{I}}\mathbf{B}, {}^{\mathsf{I}}\mathbf{R}, {}^{\mathsf{I}}\mathbf{C}, {}^{A\mathsf{I}}\mathbf{R})$.

FPD with Linear Gaussian Information State: Solution

Proposition 29 (LG FPD)

Let the system with finite-dimensional information state X_t be described by linear Gaussian (LG) model (142) and the ideal pd in FPD be also LG given by pds (143) with ${}^{AI}R > 0$. The optimal decision rule is then

$$
\begin{aligned}
{}^{O}\mathsf{f}(A_t|X_{t-1}) &= \mathcal{N}_{A_t}(\mathbf{L}_t' X_{t-1}, {}^{A}\mathbf{R}_t) \text{ with} \\
{}^{A}\mathbf{R}_t^{-1} &= {}^{AI}\mathbf{R}^{-1} + \mathbf{B}'\mathbf{S}_t^{-1}\mathbf{B} + (\mathbf{B} - {}^{I}\mathbf{B})'\, {}^{I}\mathbf{R}_t^{-1}(\mathbf{B} - {}^{I}\mathbf{B}) \\
\mathbf{L}_t' &= {}^{A}\mathbf{R}_t \left[\mathbf{B}'\mathbf{S}_t^{-1} + (\mathbf{B} - {}^{I}\mathbf{B})'\, {}^{I}\mathbf{R}^{-1}(2\mathbf{A} - {}^{I}\mathbf{A})\right]
\end{aligned} \tag{144}
$$

They are determined by positive semi-definite Riccati matrix $\mathbf{S}_t^{-1}$ that evolves

$$
\mathbf{S}_{t-1}^{-1} = \mathbf{A}\mathbf{S}_t^{-1}\mathbf{A}' + (\mathbf{A} - {}^{I}\mathbf{A})\, {}^{I}\mathbf{R}^{-1}(\mathbf{A} - {}^{I}\mathbf{A})' - \mathbf{L}_t\mathbf{R}_t\mathbf{L}_t', \ \text{with } \mathbf{S}_h^{-1} = 0. \tag{145}
$$

Proof The proof exploits directly Proposition 20 for verifying that $\gamma(X_t) \propto \mathcal{N}_{X_t}(0, \mathbf{S}_t)$, finding the decision rule and verifying the recursion for $\mathbf{S}_t^{-1}$. Detail derivation is cumbersome but straightforward and exploits (for a matrix $\mathbf{Q} > 0$ and vectors $\mathbf{n}$, x of compatible dimensions)

$$\begin{aligned} \mathsf{E}[x'\mathbf{Q}x] &= \mathsf{E}[x']\mathbf{Q}\mathsf{E}[x] + \mathrm{tr}[\mathbf{Q}\mathrm{cov}(x)] \\ x'\mathbf{Q}x + 2N'x &= (x - \hat{x})'\mathbf{Q}(x - \hat{x}) - \zeta \text{ with} \\ \hat{x} &= \mathbf{Q}^{-1}\mathbf{n}, \quad \zeta = -\hat{x}'\mathbf{Q}^{-1}\hat{x}. \end{aligned}$$

$\square$

Remarks on LQ Design

- The decision rule has the fixed form with "parameters" L_t, ${}^{A}\mathsf{R}_t$ depending on parameters of systems and on those of the ideal closed loop model. This
  - is the main source of feasibility,
  - allows to deal with time-dependent parameters, which arise in approximate evaluations (linearisation, adaptive variants with certainty-equivalence approximation).
- The same design – called LQ (linear-quadratic) – dominates the traditional design: Gaussian assumption is replaced by a quadratic performance index. The matrix ${}^{I}\mathbf{R}^{-1}$ corresponds with state penalisation and the matrix ${}^{AI}\mathbf{R}^{-1}$ action penalisation. This qualitative observation has allowed to adapt the performance index to observed behaviour: to learn the performance index.
- The term Riccati equation evolving the Riccati matrix $\mathbf{S}_t^{-1}$ is inherited from continuous time-domain.
- Numerical solution requires a significant care: LDL' type decompositions of Riccati matrix are (and should) be used.

Suboptimal Adaptive Design

- The designs with small finite $|A^\star|$, $|X^\star|$ and LG formulation are only exactly feasible cases. Generally, an approximate (suboptimal) design is needed.
- The dynamic design essentially predicts possible behaviour of the system interacting with the judged strategy and selects the most favourable one.
- The design complexity is significantly influenced by the richness of the inspected space. Its reduction is behind the majority of available approximation schemes.
- All presented approximations are connected with adaptive systems, Section 24, that approximate the optimal solution in the neighbourhood of the behaviour realisation [Kár98].
- The reader is referred to classical references [KKK95, KHB+85, Mos94, AW89] for a detailed presentation of adaptive systems.

Classification of Complexity Causes

At the design stage, the complexity stems mainly from

- complexity of the predictive pd↰ originating in complexity of processing the parametric model↰ or the observation model↰ and the time evolution model↰, which relate the knowledge↰ and the optional action↰ to the ignorance↰.
- richness of the ignorance↰ space that has to be inspected for the choice of the optimal action↰.

The suboptimal design tries to reduce the influence of one or both of these sources of complexity. The selected techniques described below are suitable to the design of the adaptive systems↰.

Approximation of Predictive Pd

- A substantial degree of the design complexity is caused by the use of predictive pds $\left(\mathsf{f}(\Delta_\tau|A_\tau,\mathcal{K}_{\tau-1})\right)_{\tau=t}^{h}$ obtained through the Bayesian filtering or estimation, Propositions 14, 15). They have the form

$$\mathsf{f}(\Delta_\tau|A_\tau,\mathcal{K}_{\tau-1}) = \int_{X_\tau^\star} \mathsf{f}(\Delta_\tau|X_\tau,A_\tau,\mathcal{K}_{\tau-1})\mathsf{f}(X_\tau|\mathcal{K}_{\tau-1})\,\mathrm{d}X_\tau. \quad (146)$$

- For a relatively short distance to the horizon h, the predictive pds (146) can be approximated by

$$\mathsf{f}(\Delta_\tau|A_\tau,\mathcal{K}_{\tau-1}) \approx \int_{X_\tau^\star} \mathsf{f}(\Delta_\tau|X_\tau,A_\tau,\mathcal{K}_{\tau-1})\hat{\mathsf{f}}_\tau(X_\tau|\mathcal{K}_{\tau-1})\,\mathrm{d}\Theta, \quad (147)$$

  with a simpler pd $\hat{\mathsf{f}}(X_\tau|\mathcal{K}_{\tau-1}) \approx \mathsf{f}(X_\tau|\mathcal{K}_{\tau-1})$.

- Approximation of time updating formula as well as usual numerical approximations of (146) (like Monte Carlo) can be interpreted as the approximation (147).

Passive Approximation

- The need to evaluate predictive pd for all possible realisations of future knowledge is a substantial source of complexity. The extreme simplification assumes that a reasonable approximation of $\mathsf{f}(X_\tau|\mathcal{K}_{\tau-1})$ is constructed from the known knowledge realisation $\mathcal{K}_{t-1}$, for all $\tau = t, \ldots, h$.
- *passive approximation* constructs

$$\hat{\mathsf{f}}(X_\tau|\mathcal{K}_{\tau-1}) = \hat{\mathsf{f}}(X_\tau|\mathcal{K}_{t-1}) \approx \mathsf{f}(X_\tau|\mathcal{K}_{\tau-1}). \tag{148}$$

- The term passive stresses the assumption that the general influence of actions on future learning is given up. The future learning and predicting are treated as if they run with the learning stopped.
- *active approximation* is any approximation, which is not passive, i.e. models influence of actions on the future learning

Certainty-Equivalence Approximation

- This approximation gets the approximate predictive pd↰ by inserting a point estimate $\hat{X}_{\tau|\tau}$ of X_τ into the observation model↰. $\hat{X}_t$

$$\mathsf{f}(\Delta_\tau|A_\tau, \mathcal{K}_{\tau-1}) \approx \mathsf{f}(\Delta_\tau|\hat{X}_{\tau|\tau}, A_\tau, \mathcal{K}_{\tau-1}). \tag{149}$$

  It corresponds to the approximation

$$\mathsf{f}(\Theta|\mathcal{K}_{\tau-1}) \approx \hat{\mathsf{f}}(X_\tau|\mathcal{K}_{\tau-1}) \equiv \delta(X_\tau - \hat{X}_{\tau|\tau}),\ \tau = t, \ldots, h, \ \text{where} \tag{150}$$

  $\delta(\cdot)$ is Dirac delta↰.
- Note that the second index of $\hat{X}_{\tau|\tau}$ stresses that the point estimate is constructed using knowledge $\mathcal{K}_{\tau-1}$, i.e. it is active approximation↰.
- The most wide spread one is passive certainty-equivalence approximation

$$\mathsf{f}(\Theta|\mathcal{K}_{\tau-1}) \approx \hat{\mathsf{f}}(X_\tau|\mathcal{K}_{\tau-1}) \equiv \delta(X_\tau - \hat{X}_{\tau|t}),\ \tau = t, \ldots, h, \ \text{where} \tag{151}$$

  $\hat{X}_{\tau|t}$ is a point estimate of X_τ based on $\mathcal{K}_{t-1}$ only.

Cautious Approximation

- The certainty-equivalence approximation ↰ works well if the pds $\mathsf{f}(X_\tau|\mathcal{K}_{\tau-1})$ or $\mathsf{f}(X_\tau|\mathcal{K}_{t-1})$ are well concentrated around $\hat{X}_{\tau|\tau}$ or $\hat{X}_{\tau|t}$.
- If there is a relatively high uncertainty about precision of the best point estimate $\hat{X}_{\tau|\tau}$, it is reasonable to include its description $C_{\tau|\tau}$ (say, covariance matrix) into the approximating pd, to use
- *cautious approximation* uses both the point estimate of unknown X_τ and a description $C_{\tau|\tau}$ of its uncertainty

$$\mathsf{f}(X_\tau|\mathcal{K}_{\tau-1}) \approx \hat{\mathsf{f}}(X_\tau|\hat{X}_{\tau|\tau}, C_{\tau|\tau}), \ \tau = t, \ldots, h. \tag{152}$$

- *super-cautious approximation* is the passive version of the cautious strategy, i.e.

$$\mathsf{f}(X_\tau|\mathcal{K}_{\tau-1}) \approx \mathsf{f}(X_\tau|\mathcal{K}_{t-1})\hat{\mathsf{f}}(X_\tau|\hat{X}_{\tau|t}, C_{\tau|t}), \ \tau = t, \ldots, h. \tag{153}$$

The name reflects the pessimism about future learning abilities.

Fight with Passivity

- Mostly, the active approximation↰s are still to complex. Thus, passive approximation↰s dominate. They are made active via the following ways.
    - An external stimulating signal is fed into the closed DM loop. It is added to optional quantities like inputs or set points. It improves learning conditions at the cost of deteriorating the achievable quality.
    - A term reflecting learning quality even under a passive-type design is added to the original loss↰ [JP72]. The design is usually numerically demanding and sensitive to the relative weight of the added term.
- Experience indicates that strategies exploiting active approximations gain just a little for design with linear models. The passivity may, however, result in completely bad performance in the case of controlled Markov-chain models [Kum85]. Systematic attempts to solve this difficult problem are rare; see reference in [FE04].

Reduction of Search Space

- We outline common techniques oriented on simplification of the optimisation space.
  Essentially, two directions can be recognised.
  - Influence of long horizon on problem complexity is suppressed.
  - The value function↰ is finitely parameterised and these parameters are estimated.
- It is worth repeating the advantage of FPD↰: instead of approximating the operation pair (minimisation,expectation) just the expectation is to be respected.

Design with a Short Horizon

- The reduction of the design horizon↰ is the most direct way to a simplified (suboptimal) design.
- The reduction obtained by planning just one-step-ahead has been popular for a long time [Pet70]. Dynamic decision making, however, means that consequences of an action↰ are encountered far behind the time moment of its application. Consequently, the action that is optimal when judged from a short-sighted perspective might be quite bad from the long-term viewpoint [KHB+85].
- This observation has stimulated the search for a compromise between the ideal planning over the whole horizon of interest and short-sighted, locally optimising strategies.

Receding Horizon (Model-Based Predictive Design)

- A little-steps-ahead planning provides just an approximation of the optimal design. Thus, it is reasonable to apply just the initial planned actions and redesign strategy whenever the knowledge about the system and its state is enriched. This is the essence of
- *receding-horizon strategy*
  - performs at time t the design looking T step ahead, with a small T bridging the dynamic consequences of the action A_t,
  - applies the first action A_t resulting from this design for the accumulated knowledge $\mathcal{K}_{t-1}$,
  - acquires the new data record $D_t = (A_t, \Delta_t)$ =(action,observation),
  - updates the knowledge $\mathcal{K}_{t-1} \rightarrow_{D_t=(A_t,\Delta_t)} \mathcal{K}_{t|t+1}$ and performs learning,
  - repeats the (design of the actions, application of the action, making the observation, performing the learning step).
- Mostly, a passive approximation of the models is used. In an extreme, widely-spread variant, known as model-based predictive design [Mos94, Pet84, CMT87, Cla94, CKBL00, STO00], it runs without the learning but coping with non-linear systems and hard bounds.

One-Step-Ahead Design Suffices with Known Value Function

- Dynamic programming, Proposition 11, and its FPD variant, Proposition 20 can be interpreted as one-step-ahead design if the value function is known. This observation justifies approximate dynamic programming [SBPW04] that estimates the value function.
- Its FPD version with a finite dimensional information state X_t has to approximate $\gamma(X_t)$ by $\gamma(\Phi, X_t)$ parameterised by a finite-dimensional parameter $\Phi \in \Phi^\star$. It should to fulfil identities

$$\begin{aligned} \gamma(\Phi, X_{t-1}) &= \int_{A_t^\star} {}^{\mathrm{I}}\mathsf{f}(A_t|X_{t-1}) \exp[-\omega(\Phi, A_t, X_{t-1})] \,\mathrm{d}A_t \qquad (154) \\ \omega(\Phi, A_t, X_{t-1}) &= \Omega(A_t, X_{t-1}) - \int_{X_t^\star} \mathsf{f}(X_t|A_t, X_{t-1}) \ln(\gamma(\Phi, X_t)) \,\mathrm{d}X_t \\ \Omega(A_t, X_{t-1}) &\equiv \int_{X_t^\star} \mathsf{f}(X_t|A_t, X_{t-1}) \ln\left(\frac{\mathsf{f}(X_t|A_t, X_{t-1})}{{}^{\mathrm{I}}\mathsf{f}(X_t|A_t, X_{t-1})}\right) \mathrm{d}X_t \end{aligned}$$

that can be conceptually solved by successive approximations while evaluating integrals by a Monte Carlo method.

Decomposition of DM

- Splitting of the DM task in a chain of subtasks is obvious and widely used way of converting optimal design to an approximation of the practically optimal design↰. The experience recommends
- *golden DM rule* , which states that a departure from the optimality should be the last option inspected.
- Below, an example of such decomposition is listed. It concerns the case of adaptive control with the learning part based on parameter estimation. Each item in the list has been found as a relatively self-containing decision sub-problem.
- Lack of the formal tools for the decomposition leaves us with empirical rules in this area. This makes us summarise here the experience we have collected in this respect within a long-term project DESIGNER [KNKP03, KH91, KH94, BKN+98].
- The design, as any human activity, is iterative. Naturally, the majority of iterations should be concentrated in the off-line phase in order to minimise expenses related to the commission of the decision strategy↰.

Off-Line Phase of Adaptive-Control Design

The following indicative list of subtasks is solved, often with hidden iterations, until the decision maker↰ is satisfied.

- Formalise the addressed DM problem, i.e.
  - get the specification of technical control aims,
  - get the specification of the system,
  - get the specification of the available data, actions and observations,
  - get the specification of technologic and complexity restrictions,
  - collect the knowledge available.
- Select class of parametric models.
- Perform experimental design allowing to collect informative data, e.g., [Zar79].
- Make data pre-processing, e.g. [OW83, KB02].
- Quantify prior knowledge [KNKP03, KKNB01, KDW+80, KBG+11].
- Estimate model structure and control period [Lju87, KNKP03, KK88, IM96, K91, BR97].

Off-Line Phase of Adaptive-Control Design (cont.)

- Estimate forgetting factor, Section 28.
- Perform generalised Bayesian estimation, Section 14, based on prior knowledge and available data; the result will be used as the prior and/or alternative pd in on-line phase, Section 28.
- Validate the model, e.g. compare quality on learning and validation data [Plu96]. The preferable solution based on Bayesian testing of hypotheses [KNŠ05] exploits all learning data and suits to dynamic systems.
- Quantify preferences via the ideal pd↰ or performance index↰, [KG10].
  Do until the results cannot be improved
  - Select the type of the suboptimal control design and its parameters.
  - Perform prior design of the controller and predict the closed-loop behaviour [KJO90, KH94, NBNT03].
  - Compare the results with decision maker's preferences.
- It is wise to store the data collected during the subsequent on-line phase and use them for an improved off-line design.

On-Line Phase of Adaptive-Control Design

The following DM subtasks are solved in real time, for $t \in t^{\star}$. Here, there is almost no freedom for iterative trial-and-error solutions.

- Collect and pre-process data.
- Generate reference signals to be followed by controlled quantities forming a part of the behaviour↰.
- Apply data-updating step and emulate time-updating step by forgetting↰.
- Use the selected suboptimal design, e.g., receding horizon strategy↰.
- Generate action using the designed strategy↰ and pre-processed data.
- Check and counteract possible discrepancies like violation of constraints (by optimised cutting) or an extreme difference of predicted and observed behaviour characteristic (by a re-tuning of optional parameters of the design↰).

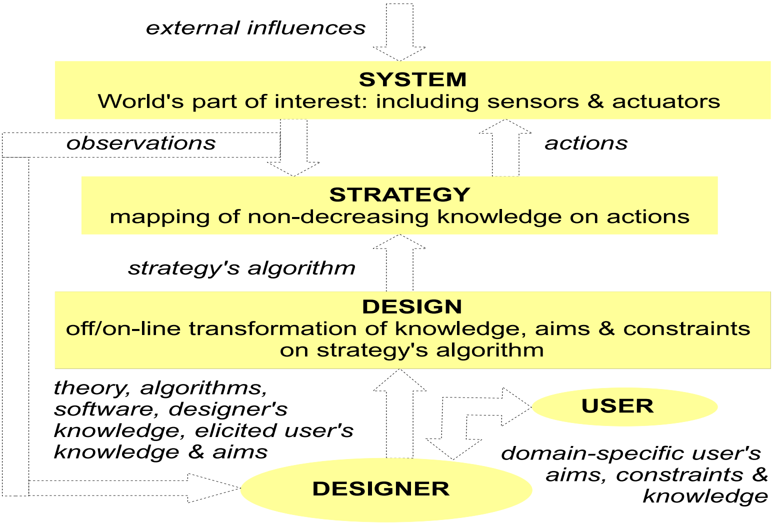
- Designs leading to linear programming [KvvPZ10].
- Handling of constrained actions [Böh04].
- General approach to learning of performance index [KG10].
- . . .

Basic Types of DM

Aim

- The additional discussion of DM elements↰, "atoms" creating any decision task, should help us to learn a good practice and to avoid common mistakes.
- The choice, modification and use DM elements↰ are specific decision subtasks that have to be harmonised with the final aim↰ considered. It is often very hard task but the golden DM rule↰ is to be respected as much as possible.
- The presentation describes the design↰ sequentially. The interrelations of respective DM elements↰ imply that design steps are mostly performed in parallel and in an iterative manner. For instance, the specification of behaviour↰ cannot be separated from the DM aim.
- The presented theory has typical versions that are over-viewed here, too.

Discussion Concerns the Bottom Part of DM Scenario



Behaviour

- DM tries to influence closed-loop behaviour in harmony with user's preferential ordering. For doing this, it has to delimit behaviour itself. It consists of a sequence up to optional horizon h. Its tth elements splits into
  - available optional action $A \in A^{\star}$
  - accessible and potentially useful observation $\Delta \in \Delta^{\star}$
  - hidden quantity $X \in X^{\star}$.
- The delimitation covers both choice of the quantity and its desired or expected range.

Actions

- Availability of an optional action↰ is inevitable precondition of any DM↰.
- The nature of DM often determines possible actions uniquely. Sometimes, alternatives are available. Such incomplete pre-determination calls for selecting the most suitable one.
- The domain-specific properties (for instance, physical or economical) and their knowledge drive the initial choice of actions. If non-unique even then, the choice should be narrowed down via Bayesian structure estimation↰, Proposition 34.

Observations

- Observations connect the artificial design world with reality.
- The causal decision rule↰ with more observations can be better than other with a more narrow observation set. Thus, no observation should be a priori discarded. Bayesian testing of hypotheses serves for the recognition whether the informational contribution of an observation entry is so small that the inevitable approximate treatment considering it is worse than without it.
- Observations splits in
  - indicators of decision quality determining preferential ordering↰
  - auxiliary, information bringing, observations: the leave to the fate↰ option is applied to them.
- *Realised* actions should be a part of observations in order to cope with implementation imprecisions. On the other hand,
- *external quantity* , which is observation↰s uninfluenced (even indirectly) by action↰, must not be taken as actions as such treatment hides the need for their prediction↰ in dynamic DM↰.

Hidden Variables

Relations describing the system of interests are modelled by exploiting two complementary and coexisting ways:

- first principles (conservation laws) that relates quantities with a clear physical meaning, which are measured indirectly by imperfect sensors. For instance, the kinematic state of a space shuttle is position, speed and acceleration while the position is imprecisely measured.
- "universal" approximation property [Hay94], i.e. the ability to approximate well any relation within a considered model class if the parameters of the approximating model are properly chosen.
  For instance, any smooth pd can be approximated by a finite mixture of Gaussian pds when weights of respective Gaussian pds and their initial two moments are appropriately chosen.

Both ways require domain knowledge and supply of universally approximating classes. None of them provides the needed complete system model and the completion has to be employed. Bayesian testing of hypotheses then serves for removal superfluous quantities.

Design Constraints

- The design↰ has to provide causal decision rule↰s. This constraint is respected by the adopted design via dynamic programming, Proposition 11. It relies on clear distinction between available observation↰s and hidden. It can be changed by judicious choice of sensors.
- Determination of ranges of measured and hidden quantities is an integral part of
- *modelling* that constructs observation model↰, time evolution model↰ and prior pd↰.
- *knowledge elicitation* is a modelling activity transforming knowledge into the prior pd↰.
  In parameter estimation↰ careful specification of quantities' ranges is often neglected and it is respected by the prior pd. It exploits fact that the posterior pd↰ has the support included in the support of the prior pd, see Section 14.

Constraints on Actions

Actions' constraints are often called

- *technological constraints* coincide with the final specification of the decision set $A^{\star}(|t^{\star}|)$. This term reflects that they follow mostly from technological, economic or safety considerations.
- Generally, non-trivial technological constraints increase complexity of the design↰. They are often relaxed for complexity reasons and reflected in other DM elements↰. Typically, the performance index↰ is modified, i.e. a sort of penalty-based optimisation [HT96] is used.
- The penalty-based coping with constraints is to be combined with a good and justified optimisation practice:
  - a simple clipping at the boundary of $A^{\star}$ should not be used if a particular unrestricted decision is out of the target set $A^{\star}$: a proper (near) optimal projection on $A^{\star}$ is needed,
  - the applied (not the designed) action is to be included into knowledge.

- Unnecessary constraints should be avoided. For instance, the restriction to unbiased estimators [Rao87a] is well justified in many statistical decision tasks. The restriction to estimators of this type in adaptive control [AW89] can make the final decision strategy (much) less efficient than possible.
- Support of the options related to constraints (similarly as the other options) indicate that a significant load is put on the designer. Algorithmically feasible support is still poorly developed.

Data Acquisition

- Data connect the artificial world of evaluations with reality. Their information content is crucial for the success of the decision making (DM↰) that use them. Ideally, their acquisition should be based on
- *experimental design* that selects strategy↰ used during data acquisition so that data are as informative as possible.
  - Theoretically, it means that the chosen working conditions should suppress ambiguity of the *best projections* caused by quality of data, see Proposition 18.
  - Practically, the optional data (system inputs, set points) have to "excite" sufficiently the inspected system↰.
    For instance, we cannot learn the dependence of outputs on inputs when inputs do not vary during the data acquisition.
  - Obviously, the excitation during the data acquisition influences also speed of the filtering↰ or estimation↰.
- The exposition focuses on cases with unknown parameter↰, i.e. time invariant hidden quantity↰ $\Theta = X_t$, $t \in t^\star$. It (almost) omits filtering↰.

On Posterior Pd

- Under natural conditions of DM, and time invariant parametric model $\mathsf{f}(\Delta|\Theta, A_t, \mathcal{K}_{t-1}) = \mathsf{M}(\Psi_t, \Theta)$ with a finite-dimensional data vector $\Psi_t' = [\Delta_t', \psi_t']$ = [observations,regression vector] the posterior pd can be expressed

$$\mathsf{f}(\Theta|\mathcal{K}_t) \propto \mathsf{f}(\Theta) \exp\left[t \int_{\Psi^\star} \frac{1}{t} \sum_{\tau=1}^{t} \delta(\Psi - \Psi_\tau) \ln(\mathsf{M}(\Psi, \Theta))\, \mathrm{d}\Psi\right]. \qquad (155)$$

- Under general conditions, the integral in (155) is bounded from above for a fixed Θ and the posterior pd asymptotically concentrates on the set containing minimisers $\Theta(\Psi^t) \in \Theta^\star$ of its negative value

$$\begin{aligned} & -\int_{\Psi^\star} \frac{1}{t} \sum_{\tau=1}^{t} \delta(\Psi - \Psi_\tau) \ln(\mathsf{M}(\Psi, \Theta(\Psi^t)))\, \mathrm{d}\Psi \qquad (156) \\ \leq\ & -\int_{\Psi^\star} \frac{1}{t} \sum_{\tau=1}^{t} \delta(\Psi - \Psi_\tau) \ln(\mathsf{M}(\Psi, \Theta))\, \mathrm{d}\Psi \end{aligned}$$

- By taking expectation of this inequality with respect to Ψ^t and Θ, we get characterisation of $\Theta(\Psi^t)$ as lower bound on

$$-\int_{\Psi^\star,\Theta^\star} \mathsf{f}(\Psi,\Theta)\ln(\mathsf{M}(\Psi,\Theta))\,d\Psi\,d\Theta \tag{157}$$

$$= \underbrace{\int_{\Psi^\star,\Theta^\star} \mathsf{f}(\Psi,\Theta)\ln\left(\frac{\mathsf{f}(\Psi)\mathsf{f}(\Theta)}{\mathsf{f}(\Psi,\Theta)}\right)d\Psi\,d\Theta}_{-\mathsf{I}(\Psi,\Theta)=-\text{mutual information}} - \underbrace{\int_{\Psi^\star} \mathsf{f}(\Psi)\ln(\mathsf{f}(\Psi))\,d\Psi}_{\text{entropy}}$$

Thus, it is desirable to minimise (157) over decision strategy used during data acquisition. As discussed with Proposition 22, entropy of data vectors has to be finite. If we fix it, the experimental design minimising (157) has to maximise

- *mutual information* between Ψ and Θ defined $\mathsf{I}(\Psi,\Theta) =$

$$\int_{\Psi^\star,\Theta^\star} \mathsf{f}(\Psi,\Theta)\ln\left(\frac{\mathsf{f}(\Psi,\Theta)}{\mathsf{f}(\Psi)\mathsf{f}(\Theta)}\right)d\Psi\,d\Theta = \mathsf{D}(\mathsf{f}(\Psi,\Theta)||\mathsf{f}(\Psi)\mathsf{f}(\Theta)). \tag{158}$$

Comments on Experimental Design

- The proposed optimisation is non-standard in this area but fits to the overall philosophy of FPD↰.
- This experimental design↰ neither supposes existence of "true" parameter nor the knowledge of the parameter $\Theta(\Psi^t) \in \Theta^\star$ describing the best projection↰ to the set of parametric models $\mathsf{M}(\Psi, \Theta)$.
- Under natural conditions of DM↰ the optimised mutual information↰ depends linearly on the the optimised strategy↰ $\{\mathsf{f}(A_t|\mathcal{K}_{t-1})\}_{t\in t^\star}$. The minimisation would led to infeasible actions without, the constrained entropy↰,
- The constraint on entropy↰ can be introduced indirectly by limiting the action range.

Comments on Experimental Design (cont.)

- The formulated optimisation is as complex as the general FPD↰ and faces the same problems as discussed in Section 29. Here, a specific simplification can be made by narrowing the set of competitive strategies even to a finite collection used within classical and well-established framework [OW83, Ogd97].
- The mutual information can also be used for analysing data when estimation↰ results are unsatisfactory.
- Feasible solutions for specific classes of models and strategies are well elaborated see, for instance, the classical reference [Zar79].

Data Pre-Processing: the Need for It

- *pre-processing* maps raw data on pre-processed data used in DM↰.
- Data pre-processing adds a dynamic mapping to the treated system↰ so that its common use seems to be illogical and harmful. However, it is fully meaningful due to the inevitable approximations in DM.
- The objective pd↰ is (practically) always out of the set considered parametric model↰s. The estimation↰ searches for the best projection↰ of the objective pd↰ that describes all relations of the considered behaviour↰ reflected in measured data.
- The projector has no information about significance of these relations with respect to the solved DM task. Data pre-processing↰ should suppress insignificant ones so that adverse influence of the additional is counteracted by the improved modelling of the important relations.

Typical Data Pre-Processing Includes:

- *data transformation* linearises non-linear data relations implied by their physical models or sensor properties. It enables use of algorithmically well-supported linear parametric model↰.
- *data scaling* realises affine data transformation. It allow standardise prior pds and decreases numerical demands.
- *outliers' suppression* removes or cuts outlying observations. It makes the system model closer to normal pd, whose processing is well-supported but which is non-robust in presence of outliers.
- *noise suppression* removes data constituents, typically of a high frequency, reflecting more sensor behaviour than the system dynamics.
- *missing data treatment* substitutes missing data by their guess. It counteracts lack of informative data and fills the gap in time sequences reflecting the system dynamics.
- *re-sampling* standardises sampling rate of the preprocessed data. It exploits a high frequency of the data acquisition for noise suppression and removes sampling-induced variations of modelled relations.

Common Pre-Processing Mistakes

- Pre-processing significantly influences quality of the resulting projection of the objective pd on the set of parametric models and thus whole DM.
- Damages made in the pre-processing phase can hardly be removed in later design phases. Typical errors in pre-processing are:
  - a premature reduction of data leading to a loss of relevant informative,
  - wasting of information due to the too low-frequency sampling of acquired data,
  - a significant change of the modelled dynamics by the pre-processing block: for instance, introduction too high transportation delay,
  - a distortion of the inspected relations by inadequate substitutions for missing data.

On Pre-Processing Methodology

Problem 1 (How to Harmonise Pre-Processing with Ultimate Goal?)

Similarly as other sub-tasks, the optimal pre-processing requires solution of the overall decision task to which it serves. It is mostly impossible. Even splitting the overall task into adequate and harmonised subtasks is left to a "sound" reasoning. It is pleasant as it requires creativity. It is unpleasant as the final results of the decision making might be spoiled by an improper choice. The problem is severe especially in dynamic design↰ in which there is a restricted freedom for an iterative trial-and-error treatment.

Construction of Parametric Model – Grey Box Modelling

The parametric model↰ relates behaviour↰'s constituents and provides predictive pd↰ exploited in data-driven design↰. A substantial domain-specific knowledge should be built into it. The ideally, it should be based on

- *grey box* modelling collects theoretically expected relations between quantities in behaviour↰ and extend them into the probabilistic parametric model↰.
  Parameters are unknown constants (almost always) present in the final parametric model↰.
- The extension is to be done using minimum KLD principle↰.
- The resulting parametric model↰ is usually too complex for subsequent treatment and is to approximated by a member of a feasible family, typically, dynamic exponential family↰. The approximation principle↰ discussed in Section 20 serves to this purpose.
- The dynamic exponential family↰ is a natural candidate as approximating pd, which converts functional recursive estimation into algebraic one , see Proposition 27.

Black Box Modelling

- The grey box modelling can be either impossible due to the lack of domain knowledge or can lead to unmanageable models. Then
- *black box* modelling approximates the modelled functional relations by expanding them into a suitable functional basis. The functional basis is required to be dense within the class of modelled mappings. Neural-nets community [Hay94] casted for it the appealing term
- *universal approximation property* , which means the ability of a function class ("basis") to approximate arbitrarily well any modelled function.

  traditional applied to moments; general leads to mixture models

Traditionally, the expansion is applied to moments of the approximated pd. Obviously, an expansion of the pd itself is more complete and systematic and sometimes even simpler [JU04].

Modelling by Finite Mixtures

This prominent black box↰ model is discussed in connection with modelling of a data vector↰ $\Psi'_t = [\Delta'_t, \psi'_t]$ =[observation↰,regression vector↰].

- *finite mixture* is parametric pd of the form

$$\mathsf{f}(\Psi_t|\Theta) = \sum_{c \in c^\star} \alpha_c \mathsf{f}(\Psi_t|\Theta_c), \; c^\star = \{1, \ldots, |c^\star|\}, \; |c^\star| < \infty, \text{ given by } \quad (159)$$

- *component* , the pd $\mathsf{f}(\Psi_t|\Theta_c)$, which is typically (not inevitably) a member of exponential family↰, and component parameter Θ_c
- *component weight* α_c, whose collection $\alpha = (\alpha_c)_{c \in c^\star}$ has properties of pd of an unobserved discrete-valued
- *pointer to the component* $C_t \in c^\star$, $\mathsf{f}(C_t = c|\Theta) = \alpha_c$. C_t can be interpreted as an hidden quantity↰ within the modelled part of the behaviour↰

$$(C_t, \Theta = (\Theta_c, \alpha_c)_{c \in c^\star}, \text{observed quantities forming } \Psi_t) \qquad (160)$$

Universal Approximation Property of Finite Mixtures

- Let pd $\mathsf{f}(\Psi)$ be ν measurable with compact Hausdorff domain [Bou66]. Then, it can be approximated by a piece-wise function

$$\mathsf{f}(\Psi) \approx \sum_{c=1}^{\infty} \mathsf{f}(\tilde{\Psi}_c)\mathrm{vol}_\mathrm{c} \frac{\chi_\mathrm{c}(\Psi)}{\mathrm{vol}_\mathrm{c}}, \tag{161}$$

  where $\chi_i(\Psi)$ is a small neighbourhood of the grid point $\tilde{\Psi}_c$ with volume $\mathrm{vol}_\mathrm{c} = \int_{\Psi^\star} \chi_\mathrm{c}(\Psi)\,\mathrm{d}\Psi$. This is countable mixture of uniform pdfs. Their non-negative weights $\mathsf{f}(\tilde{\Psi}_c)\mathrm{vol}_\mathrm{c}$ have to fall to zero as $\int \mathsf{f}(\Psi)_{\Psi^\star}\mathsf{f}(\Psi)\,\mathrm{d}\Psi = \int \mathsf{f}(\Psi)_{\Psi^\star} \frac{\chi_c(\Psi)}{\mathrm{vol}_\mathrm{c}}\,\mathrm{d}\Psi = 1$, $c \in c^\star$. Thus, $\mathsf{f}(\Psi)$ can be approximated arbitrarily well by a finite mixture of uniform pds.
- Indicators of the explored decomposition of unity [Vla79] can be approximated by other, even infinitely smooth non-negative functions having finite integral. They provide other basis for creating mixtures and allow to relax compactness assumption.

giwmixapproxte

Finite Mixtures on Entire Behaviour: Independent Case

- The mixture model (159) describes only a part of the behaviour. Additional assumptions are needed to get its complete parametric description. The wide-spread modelling deals with
- *classic mixture* , which assumes data vectors Ψ_t independent when conditioned on Θ (160), [TSM85]. The corresponding likelihood↰

$$\mathsf{L}(\Theta, \mathcal{K}_t) \equiv \prod_{\tau \le t} \mathsf{f}(\Psi_\tau | \Theta) = \prod_{\tau \le t} \sum_{c \in c^\star} \alpha_c \mathsf{f}(\Psi_\tau | \Theta_c) \qquad (162)$$

is the sum of 2^t different functions of Θ.
Formula (162) demonstrates extreme complexity of the exact Bayesian estimation↰. The induced estimation↰ complexity is faced by several ways commented below.

Finite Mixtures on Entire Behaviour: Dependent Case

The finite mixture (159) generally induces the parametric model

$$f(\Delta_t|\Theta,\psi_t) \equiv M(\Psi,\Theta) = \frac{\sum_{c\in c^\star} \alpha_c f(\Psi_t|\Theta_c)}{\sum_{c\in c^\star} \alpha_c \underbrace{\int_{\Delta_t^\star} f(\Psi_t|\Theta_c)\,\mathrm{d}\Delta_t}_{f(\psi_t|\Theta_c)}} \quad (163)$$

$$= \sum_{c\in c^\star} \beta_c(\Theta,\psi_t) \underbrace{\frac{f(\Psi_t|\Theta_c)}{f(\psi_t|\Theta_c)}}_{f(\Delta_t|\Theta_c,\psi_t)} = \sum_{c\in c^\star} \beta_c(\Theta,\psi_t) f(\Delta_t|\Theta_c,\psi_t),$$

$$\beta_c(\Theta,\psi_t) = \alpha_c \frac{f(\psi_t|\Theta_c)}{\sum_{c\in c^\star} f(\psi_t|\Theta_c)}, \Psi_t = \overbrace{[\Delta_t', \underbrace{\psi_t'}_{\textit{regression vector}}]}^{\textit{data vector}}$$

i.e. the finite mixture has universal approximation property even in the dependent case if the components weights $\beta_c(\Theta,\psi_t)$ are allowed to depend on the regression vector ψ_t. In fact the model (163) is ratio of coupled finite mixtures.

Complexity and Established Ways of Coping with It

The *exact estimation and prediction* with the mixture model is practically impossible even in independent case as the number of terms in the likelihood $\mathsf{L}(\Theta, \mathcal{K}_t)$ (162) increases exponentially. Good approximations exist if components belong to exponential family.
The available techniques clusters into the following groups, which are also used for other dependence models.

- Search for point estimates maximising likelihood, typically, via expectation-maximisation algorithm [DLR77],
- Approximation of the intractable prior pd by the product of approximate prior conjugate pds to respective components and by heuristic (quasi-Bayesian) [TSM85, KKS98, KBG$^+$06] or KLD-based projection (85) [And05] of the posterior pd into the product class.
- Formulation of the learning problem as filtering, which explicitly estimates realisation of the pointer to the component C_t.

Prior Pd

- The need for selecting prior pd is often regarded as the main disadvantage of the adopted Bayesian approach. The lack of efficient, unambiguous and elicitation-expert independent [GKO05], tools for knowledge elicitation can be blamed for it.
- Here, we contribute positively to the never-ending discussion on pros and cons of exploiting prior pds by indicating that the prior "expert" information can be introduced into learning in a systematic way.
- The posterior pd (54) is a product of the likelihood consisting of t factors coinciding with parametric models $\mathsf{f}(\Delta_\tau|\Theta, A_\tau, \mathcal{K}_{\tau-1})$ and of a single prior pd $\mathsf{f}(\Theta)$. If t is high enough and data bring a sufficient information on Θ then the posterior pds obtained for various prior pds resemble each other: the role of prior pd is weak [DeG70].
- The posterior pd is significantly influenced by the prior pd when some of the above conditions is not fulfilled.

Qualitative Role of Prior Pd

Proposition 30 (Role of the Prior Pd)

- *Parameter values* $\Theta \notin \mathrm{supp}\,[\,\mathsf{f}(\Theta)]$, *for which the prior pd is zero, get the zero posterior pd, too. Formally,*

$$\mathrm{supp}\,[\,\mathsf{f}(\Theta|\mathcal{K}_t)] = \mathrm{supp}\,[\,\mathsf{L}(\Theta, \mathcal{K}_t)] \cap \mathrm{supp}\,[\,\mathsf{f}(\Theta)]\,.$$

- *The recursive evolution of the likelihood*

$$\begin{aligned} \mathsf{L}(\Theta, \mathcal{K}_t) \quad &\equiv \quad \prod_{\tau \le t} \mathsf{f}(\Delta_\tau|\Theta, A_\tau, \mathcal{K}_{\tau-1}) = \mathsf{f}(\Delta_t|\Theta, A_t, \mathcal{K}_{t-1})\mathsf{L}(\Theta, \mathcal{K}_{t-1}) \\ & \qquad t \in t^\star,\ \mathsf{L}(\Theta, \mathcal{K}_0) = 1, \Theta \in X^\star_{\mathsf{L}} \end{aligned} \tag{164}$$

does not depend on the prior pd chosen.

- *The posterior pd exists iff the product* $\mathsf{L}(\Theta, \mathcal{K}_t)\mathsf{f}(\Theta)$ *is integrable.*

Proof It is a direct consequence of the formula for posterior pd (54). □

Remarks on the Role of Prior Pd

Remark 18

- *The prior pd offers a simple and clear way for introducing hard restrictions on parameters.*
- *The recursion (164) is valid even if* $\Theta^*_{\mathsf{L}} \neq \Theta^\star \equiv \mathrm{supp}\,[\,\mathsf{f}(\Theta)]$. *This implies that hard bounds on parameter values must not influence likelihood*↰. *This is repeatedly overlooked in recursive estimation. Instead of restricting the posterior pd*↰, *the likelihood statistics are deformed with an adverse effect on the estimation quality.*
- *Often, a flat prior pd models the lack of prior knowledge. Even integrability of the prior pd is relaxed and the*
- *improper prior pd* $\mathsf{f}(\cdot) \geq 0, \; \int \mathsf{f}(\Theta)\, \mathrm{d}\Theta = \infty$ *are used. For instance, the posterior pd is proportional to the likelihood if we allow the prior pd be improper and equal 1. Then, the posterior pd*↰ *might be improper, too, i.e. a flat but proper prior pd regularises estimation.*

Babel Tower Problem

Automatic mapping of many different forms of processed knowledge calls for a common language for expressing the knowledge irrespectively of the form of the parametric model↰. It is provided by

- *fictitious data* is a possible outcome gedanken experiment on the modelled system. The following examples indicate suitability of this knowledge expression but its universality is only conjectured.
- *obsolete data* covering data measured on a similar or simulated system or expected data ranges motivated physically. Experimentally motivated characteristics like:
- *static gain* is (approximately) constant steady-state output value observed after applying unit step on input.
- *step response* is time response of the output on the unit input step.
- *frequency characteristic* at frequency ω is an expected form of the output $a(\omega)\sin(t\omega + \phi(\omega))$ after applying the input $\sin(\omega t)$ ($a(\omega)$ is called amplitude and $\phi(\omega)$ phase).
- *cut off frequency* is frequency ω_c after which $a(\omega) \approx 0$.

The estimation with fictitious data in the role of observations provides an approximate posterior pd as the fictitious data does not come from the modelled system in its present form. It leads to the analogous situation as in forgetting, Section 28, where the approximate nature of the estimation originated in parameter changes. This motivates the same formulation.

Proposition 31 (Coping with Imprecise Estimation)

Let f *be an unknown pd,* f_0 *its prior guess complemented by the knowledge* $\mathsf{D}(\mathsf{f}||\hat{\mathsf{f}}) \leq \beta$*, i.e. a given* $\hat{\mathsf{f}}$ *approximates* f*, cf. (85), with the precision not exceeding a given* $\beta > 0$*. Then, the minimum entropy principle* *recommends to use*

$$\mathsf{f} \propto \hat{\mathsf{f}}^{\lambda}\mathsf{f}_0^{1-\lambda}, \text{ where } \lambda = \begin{cases} 1 & \text{if } \mathsf{D}(\mathsf{f}_0||\hat{\mathsf{f}}) < \beta \\ \in (0,1) & \text{otherwise} \end{cases} \tag{165}$$

Proof The Lagrangian $\mathsf{D}(\mathsf{f}||\mathsf{f}_0) + \Lambda\mathsf{D}(\mathsf{f}||\hat{\mathsf{f}})$ with Kuhn-Tucker multiplier $\Lambda \geq 0$ is just rearranged into KLD of f on the claimed solution with active and non-active constraint. Then, Proposition 17, is used. □

Fictitious Time, Pre-Prior and Pre-Posterior Pds

- Sequences of fictitious data are sought to be realised *before* processing real data, the timed quantity↰ labelled by discrete time $t \in t^{\star}$. To distinguish them, we use
- *fictitious time* , $k \in \{1, 2, \ldots |k^{\star}|\}$, $|k^{\star}| < \infty$, labels individual items of sequences fictitious data↰.
- Bayesian estimation↰ is applied to fictitious data (with some modifications discussed further on). It starts from
- *pre-prior pd* , which is (usually flat) pd $\bar{\mathsf{f}}$ delimiting expected range $\Theta^{\star}$ of unknown parameter↰ Θ and leads to
- *pre-posterior pd* , which is the posterior pd obtained by (modified) Bayesian estimation↰ applied to pre-prior pd and fictitious data. It becomes prior pd↰ after processing all fictitious data.

Estimation with Fictitious Data I

The fictitious data update the pd $\mathsf{f}(\Theta|\mathcal{K}_0) = \mathsf{f}_0(\Theta)$ into the pre-posterior pd

$$\hat{\mathsf{f}}(\Theta) = \hat{\mathsf{F}}(\Theta|\mathcal{K}_k) \propto \mathsf{f}(\Delta_k|\Theta, A_k, K_{k-1})\mathsf{f}(\Theta|\mathcal{K}_{k-1}). \tag{166}$$

Essence of the fictitious data implies that the $\hat{\mathsf{f}}(\Theta)$ approximates the unknown correct pd $\mathsf{f}(\Theta|\mathcal{K}_k) = \mathsf{f}(\Theta)$, which would express properly the processed knowledge piece. The used simplified identifiers connect the respective pds with those in Proposition 31, which recommends to take the following pd as the correct one

$$\mathsf{f}(\Theta|\mathcal{K}_k) \propto \mathsf{f}^{\lambda_k}(\Delta_k|\Theta, A_k, K_{k-1})\mathsf{f}(\Theta|\mathcal{K}_{k-1}),\ \lambda_k \in (0,1],\ \text{i.e.} \tag{167}$$

Fictitious data in Bayes rule is to enter the "flattened" parametric model.

It means that adequate processing of the fictitious data uses

- *weighted Bayes rule* , which processes (fictitious) data by (167).

Estimation with Fictitious Data II

A data vector Ψ_k determines a used parametric model $\mathsf{f}(\Delta_k|\Theta, A_k, K_k) = \mathsf{M}(\Psi_k, \Theta)$, $\Psi'_k = [\Delta_k, \psi'_k]$ = fictitious [observation, regression vector]. (168)

The weighted Bayes rule (167), applied to pre-prior pd $\bar{\mathsf{f}}(\Theta)$, gives

$$\mathsf{f}(\Theta|\mathcal{K}_{|k^\star|}) \propto \bar{\mathsf{f}}(\Theta) \exp\left[w_\kappa \sum_{k=1}^{|k^\star|} \alpha_k \ln(\mathsf{M}(\Psi_k, \Theta))\right] \tag{169}$$

$$= \bar{\mathsf{f}}(\Theta) \exp\left[w_{|k^\star|} \int_{\Psi^\star} \mathsf{f}_{|k^\star|}(\Psi) \ln(\mathsf{M}(\Psi, \Theta))\, \mathrm{d}\Psi\right], \quad \text{where}$$

$$w_{|k^\star|} = \sum_{k=1}^{|k^\star|} \lambda_k \in (0, |k^\star|], \ \alpha_k = \frac{\lambda_k}{w_{|k^\star|}} \in (0, 1] \ \Rightarrow \sum_{k=1}^{|k^\star|} \alpha_k = 1$$

$$\mathsf{f}_{|k^\star|}(\Psi) = \sum_{k=1}^{|k^\star|} \alpha_k \delta(\Psi - \Psi_k),$$

where δ denotes Dirac delta and $\mathsf{f}_{|k^\star|}(\Psi)$ can be interpreted as a weighted version of the sample pd on the (fictitious) data vector Ψ, cf. (155).

Grouping of Fictitious Data

- The quality of knowledge elicitation, formally performed by the weighted Bayes rule↰, depends strongly on the chosen weights λ_k that are determined by the chosen precisions β_k (165).
- The numbers of fictitious data↰ expressing knowledge pieces having different sources (different obsolete data, different, physical aspects, different experts' opinion) may differ substantially.
  These observations motivate to group fictitious data by defining
- *homogenous knowledge piece* , indexed by $\kappa \in \kappa^\star = \{1, 2, \ldots, |\kappa^\star|\},\ |\kappa^\star| < \infty$, which is expressed by fictitious data Ψ_k, $k \in k^\star_\kappa \subset k^\star$ with a common weight $\lambda_k = \lambda_\kappa$ in (169).

Note that the number of fictitious data $|k^\star|$ can be very large (even infinite) as the fictitious data↰ can result from analytically performed gedanken experiment or may result from an extensive simulations. The number of homogenous knowledge pieces $|\kappa^\star|$ is always finite.

- Considering all homogenous knowledge pieces, the pre-posterior pd (169) becomes the prior pd of the form

$$\begin{aligned} \mathsf{f}(\Theta) &\propto \bar{\mathsf{f}}(\Theta)\exp\left[\sum_{\kappa\in\kappa^\star}\nu_\kappa\int_{\Psi^\star}\mathsf{f}_\kappa(\Psi)\ln(\mathsf{M}(\Psi,\Theta))\,\mathrm{d}\Psi\right] \\ \nu_\kappa &\in (0,\infty) \text{ and } \mathsf{f}_\kappa(\Psi) \text{ is pd on } \Psi^\star. \end{aligned} \tag{170}$$

- The pd $\mathsf{f}_\kappa(\Psi)$ is formally $\mathsf{f}_\kappa(\Psi) = \frac{1}{|k_\kappa{}^\star|}\sum_{k\in k_\kappa^\star}\delta(\Psi-\Psi_k)$. Its weight $\nu_\kappa = |k_\kappa{}^\star|\lambda_\kappa \leq |k_\kappa{}^\star|$ as $\lambda_k \in (0,1]$, see (167). The number $|k_\kappa{}^\star|$ can be infinite. The limited knowledge precision implies $\nu_\kappa < \infty$.
- The formula (170) was proposed and discussed in [KAB$^+$06].

Compact Form of Knowledge Elicitation

- The choice of the pre-prior pd in the form mimic to a knowledge pieces

$$\bar{\mathsf{f}}(\Theta) \propto \exp\left[\bar{\nu}\int_{\Psi^\star} \bar{\mathsf{f}}(\Psi)\ln(\mathsf{M}(\Psi,\Theta))\,\mathrm{d}\Psi\right], \tag{171}$$

given by $\bar{\nu} \geq 0$ and a pd $\bar{\mathsf{f}}(\Psi)$, leads to the compact form of the prior pd

$$\begin{aligned} \mathsf{f}(\Theta) &\propto \exp\left[\nu_0\int_{\Psi^\star} \mathsf{f}_0(\Psi)\ln(\mathsf{M}(\Psi,\Theta))\,\mathrm{d}\Psi\right] \qquad (172)\\ \nu_0 &= \bar{\nu} + \sum_{\kappa\in\kappa^\star}\nu_\kappa \in (0,\infty) \text{ and } \mathsf{f}_0(\Psi) = \frac{\bar{\nu}\bar{\mathsf{f}}(\Psi) + \sum_{\kappa\in\kappa^\star}\nu_\kappa \mathsf{f}_\kappa(\Psi)}{\bar{\nu} + \sum_{\kappa\in\kappa^\star}\nu_\kappa}, \end{aligned}$$

i.e. the pd $\mathsf{f}_0(\Psi)$ is obtained by merging (100) (with appropriate change of notation) of pds representing respective knowledge pieces.

Knowledge Elicitation in Exponential Family

- For dynamic exponential family↰ $\mathsf{f}(\Theta)$ becomes conjugate prior↰ (118)

$$\begin{aligned} \mathsf{f}(\Theta) &\propto \bar{\mathsf{f}}(\Theta)\mathsf{A}^{\nu_0}(\Theta)\exp\langle V_0, \mathsf{C}(\Theta)\rangle \\ \nu_0 &= \bar{\nu} + \sum_{\kappa\in\kappa^\star} \nu_\kappa, \;\; V_0 = \bar{\nu}\bar{V} + \sum_{\kappa\in\kappa^\star} \nu_\kappa V_\kappa \\ V_\kappa &= \int_{\Psi^\star} \mathsf{B}(\Psi)\mathsf{f}_\kappa(\Psi)\,\mathrm{d}\Psi, \;\; \kappa \in \kappa^\star, \end{aligned} \tag{173}$$

  and κth homogenous knowledge piece describes expectation of $\mathsf{B}(\Psi)$, i.e. it gets the form of generalised moments of Ψ.

- Practical examples of transformation knowledge pieces in fictitious data are in [KN00, KBG+11].

The Choice of Homogenous Groups and Their Weights

- Homogenous groups are mostly implied by meaning of the processed knowledge. Their specification does not seem problematic.
- The weights ν_κ can be chosen subjectively to reflect reliability of the knowledge source. This is, however, dangerous as
  - reliability and its guess have high volatility,
  - knowledge pieces can be mutually dependent, even repeated.
- Thus, it is desirable to choose the weights more objectively. Practically, it can be done if some observed data reflecting the current state of the modelled system are available for the choice of weights.

Dependence of Predictive Pd on Weights

- For a given data d^t, and any non-negative weights $\nu^{|\kappa^\star|}$, the predictive pd has the value, see (57),

$$\begin{aligned} f(d^t|\nu^{|\kappa^\star|}) &= \frac{J(\nu_t, f_t)}{J(\nu_0, f_0)}, \quad \text{where} \qquad (174) \\ J(\nu, f) \quad &\text{is} \quad \text{normalisation factor (57)} \end{aligned}$$

given by the parametric model $M(\Psi, \Theta)$ (155) with arguments

$$\begin{aligned} \nu_0 &= \bar{\nu} + \sum_{\kappa \in \kappa^\star} \nu_\kappa, \text{ see (172)}, \ \nu_t = \nu_0 + t \\ f_0 &= f_0(\Psi) = \frac{\bar{\nu}}{\nu_0} \bar{f}(\Psi) + \sum_{\kappa \in \kappa^\star} \frac{\nu_\kappa}{\nu_0} f_\kappa(\Psi), \text{ see (172)}, \\ f_t &= f_t(\Psi) = \frac{\nu_0}{\nu_t} f_0(\Psi) + \frac{t}{\nu_t} \frac{1}{t} \sum_{\tau=1}^{t} \delta(\Psi - \Psi_\tau). \end{aligned}$$

Data vectors Ψ_τ are made of the observed data d^t, δ is Dirac delta.

Data-Based Choice of Weights

The choice of ν is our (designer's') action, which should make the predictive pd good approximation of the available sample pd. The recommended minimisation of KLD (85) reduces to maximisation of the predictive pd (174) with respect to $\nu^{|\kappa^\star|}$ with entries bounded by $H < \infty$.

- Inspection of the normalisation factor J (57), given by the parametric model M(Ψ, Θ) (155) helps in judging complexity of this problem. It holds

$$\mathsf{J}(\nu, \mathsf{f}) = \int_{\Theta^\star} \exp\left[\nu \int_{\Psi^\star} \mathsf{f}(\Psi) \ln(\mathsf{M}(\Psi, \Theta))\, \mathrm{d}\Psi\right] \mathrm{d}\Theta \qquad (175)$$

- Non-trivial pre-prior pd makes both $\mathsf{J}(\nu_\tau, \mathsf{f}_\tau)$, $\tau \in \{0, t\}$ finite for any $\nu^{|\kappa^\star|} \in [0, H]^{|\kappa^\star|}$, $H < \infty$.
- $\nu^{|\kappa^\star|}$ enters exponent in (175) linearly and thus J is continuous convex function of $\nu^{|\kappa^\star|}$.

Data-Based Choice of Weights (cont.) & Open Problems

- The predictive pd (174) is the ratio of continuous, convex positive functions of $\nu^{|\kappa^{\star}|}$ and reaches its maximum on $|\kappa^{\star}| < \infty$ dimensional compact. Thus, the maximisation of the predictive pd has nontrivial solution achievable by standard algorithms, for instance, [Ben06].
- The choice of the upper bound H does not seem critical. The admission of zero weights is important as it allows to suppress repetitively presented prior knowledge contradicting with observations.

Problem 2 (Problems Related to the Elicitation)

- *Does exist a significant class of prior knowledge that cannot be expressed via fictitious data?*
- *Applicability of the same methodology to filtering is conjectured but untried.*
- *Applicability to preference elicitation is conjectured but untried.*

Preference Description

- The discussion focuses on the ideal pd ${}^{\mathsf{I}}\mathsf{f}(B)$ as a descriptor of preferential ordering within FPD. It is general enough as any Bayesian DM with strategy-independent performance index (19) $\mathsf{I}(B) = \mathsf{I}_{\mathsf{S}}(B)$ can be converted into the ideal pd using Proposition 23

$$ {}^{\mathsf{I}}\mathsf{f}(B) \quad = \quad \frac{\mathsf{M}(B)\exp\left[-\mathsf{I}(B)/\lambda\right]}{\int_{B^\star}\mathsf{M}(B)\exp\left[-\mathsf{I}(B)/\lambda\right]\,\mathrm{d}B}, \; \lambda > 0, \; \lambda \approx 0, \quad (176) $$

  where $\mathsf{M}(B)$ is system model recognised in factorisation (50) of the closed loop model $\mathsf{f}_{\mathsf{S}}(B) = \mathsf{M}(B)\mathsf{S}(B)$.

- A preferential quantity ${}^{\mathsf{I}}X$, a hidden quantity introduced in order to get complete ordering of behaviours $B^\star$, splits the hidden quantity X

$$ X = ({}^{\mathsf{M}}X, {}^{\mathsf{I}}X), \qquad (177) $$

  where ${}^{\mathsf{M}}X$ enters explicitly system model $\mathsf{M}(B)$ and generally the ideal pd ${}^{\mathsf{I}}\mathsf{f}(B)$ while ${}^{\mathsf{I}}X$ enter the ideal pd ${}^{\mathsf{I}}\mathsf{f}(B)$ only.

The Choice of the Set of Ideal Pds

The rational choice of the set of possible ideal pds ${}^{\mathsf{I}}\mathsf{f}^{\star}$ is to respect that

- the ideal pd ≡ ${}^{\mathsf{I}}\mathsf{f} \equiv \mathsf{f}_{{}^{\mathsf{I}}\mathsf{S}} =$ closed loop model with the optimal strategy ${}^{\mathsf{I}}\mathsf{S}$ minimising expected performance index I.
- the ideal pd in the standard Bayesian design is the system model multiplied by the factor $\exp[-\mathsf{I}(B)/\lambda]$ (176) that can be interpreted as an indicator of the set of desired behaviours $B_{\star} \subset B^{\star}$.

Altogether, the ideal pd should resemble system model restricted to $B_{\star}$.
This induces rules like:

- Popular quadratic performance index, which corresponds to normal closed loop model, should be used when (approximate) normality is reachable: it does not suit to systems described by heavy-tailed pds.
- Support of the ideal pd is to be the desirable $B_{\star}$, e.g., admission of large actions' variances can make the optimal strategy useless.
- The decision horizon in approximate design like receding-horizon strategy has to respect closed-loop dynamics as a chaining of short horizon optimisation may lead to poor performance [KHB+85].

Algorithmic Preference Elicitation

TO BE COMPLETED, SEE ATLANTA, WINDSURFER, SIERRA NEVADA

Static Tasks Differ in Generality

- The static design
  - selects and uses a single decision rule
  - does not check dynamic consequences of the action taken.
- Categories of static design are distinguished according to content of ignorance.
- The presented classical examples indicate how to formalise DM tasks.

General Static Design

Setting the action time $t = 1$, the static design needs the DM elements↰:

- behaviour↰ $B=(\mathcal{G}_A,A,\mathcal{K}_A)=$(ignorance↰,action↰,knowledge↰) $=$((hidden,unmade observations),action,knowledge) $= ((X = X_1, X_0), \Delta = \Delta_1), A = A_1, \mathcal{K}_A = \mathcal{K}_0)$
- admissible decision rule↰s meeting (45) $\mathsf{S}(A|X_0, \mathcal{K}_0) = \mathsf{S}(A|\mathcal{K}_0)$
- observation model↰ $\mathsf{f}(\Delta|X, A, \mathcal{K}_0)$ & its ideal $^{\mathsf{I}}\mathsf{f}(\Delta|X, A, \mathcal{K}_0)$
- time evolution model↰ $\mathsf{f}(X|X_0, A, \mathcal{K}_0)$ & its ideal $^{\mathsf{I}}\mathsf{f}(X|X_0, A, \mathcal{K}_0)$
- prior pd↰ $\mathsf{f}(X_0|\mathcal{K}_0)\mathsf{f}(\mathcal{K}_0)$ & its ideal $^{\mathsf{I}}\mathsf{f}(X_0, \mathcal{K}_0)$
- ideal decision rule↰ $^{\mathsf{I}}\mathsf{S}(A|X_0, \mathcal{K}_0)$.

Notice: unchecked ignorance is influenced by the action made!

One-Step-Ahead Prediction

Example 12 (One-Step-Ahead Prediction)

aim	*to construct prediction* $\hat{\Delta} \in \Delta^\star$ *of unmade observation* $\Delta = \Delta_1 \in \Delta^\star$ *modelled by observation model* $$\mathsf{f}(\Delta \mid X_1, \mathcal{K}) = \mathsf{f}(\Delta \mid X_1, \hat{\Delta}, \mathcal{K}) \qquad (178)$$ *and time evolution model* $$\mathsf{f}(X_1 \mid X_0, \mathcal{K}) = \mathsf{f}(X_1 \mid X_0, \hat{\Delta}, \mathcal{K}) \qquad (179)$$
system	*modelled World part*
action	$\hat{\Delta} \in \Delta^\star$
knowledge	$\mathcal{K}$ *entering both models and prior pd* $\mathsf{f}(X_0 \mid \mathcal{K})$
ignorance	*hidden* X_1, X_0 *and observation* Δ
uncertainty	*anything preventing to determine fully* X_1, X_0, Δ *from* $\mathcal{K}$
constraint	$\Delta^\star$, *computational complexity*
dynamics	*none (single decision rule is required)*

One-Step-Ahead Prediction: DM Elements

DM elements↰ relevant to the one-step-ahead prediction are

- behaviour↰ $B{=}(\mathcal{G}_A,A,\mathcal{K}_A){=}$(ignorance↰,action↰,knowledge↰)
  =((unmade observations,hidden),prediction,knowledge)
  $= ((\Delta = \Delta_1, X_1, X_0), \hat{\Delta}, \mathcal{K}_{\hat{\Delta}} = \mathcal{K})$
- admissible decision rule↰s meeting (45) $\mathsf{S}(\hat{\Delta}|X_1, X_0, \mathcal{K}) = \mathsf{S}(\hat{\Delta}|\mathcal{K})$
- observation model↰ $\mathsf{f}(\Delta|X_1, \hat{\Delta}, \mathcal{K}) = \mathsf{f}(\Delta|X_1, \mathcal{K})$
- time evolution model↰ $\mathsf{f}(X_1|X_0, \hat{\Delta}, \mathcal{K}) = \mathsf{f}(X_1|X_0, \mathcal{K})$
- prior pd↰ $\mathsf{f}(X_0|\mathcal{K})\mathsf{f}(\mathcal{K})$
- the ideal pd

$$^{\mathsf{I}}\mathsf{f}(\Delta, \hat{\Delta}, X_1, X_0, \mathcal{K}) = {}^{\mathsf{I}}\mathsf{f}(X_1, X_0|\Delta, \hat{\Delta}, \mathcal{K})\, {}^{\mathsf{I}}\mathsf{f}(\Delta|\hat{\Delta}, \mathcal{K})\, {}^{\mathsf{I}}\mathsf{S}(\hat{\Delta}|\mathcal{K})\, {}^{\mathsf{I}}\mathsf{f}(\mathcal{K}) \qquad (180)$$

One-Step-Ahead Prediction: the Choice of Ideal Pd

The chosen factorisation makes options of factors in ideal pd "natural"

$$^{\mathsf{I}}\mathsf{f}(X_1, X_0|\Delta, \hat{\Delta}, \mathcal{K}) = \mathsf{f}(X_1, X_0|\Delta = \hat{\Delta}, \mathcal{K}) \tag{181}$$

$\Leftrightarrow$ knowledge in $\hat{\Delta}, \mathcal{K}$ ideally coincides with knowledge in $\Delta, \hat{\Delta}, \mathcal{K}$

$$^{\mathsf{I}}\mathsf{f}(\Delta|\hat{\Delta}, \mathcal{K})\, ^{\mathsf{I}}\mathsf{f}(\mathcal{K}) = \, ^{\mathsf{I}}\mathsf{f}(\Delta|\mathcal{K})\, ^{\mathsf{I}}\mathsf{f}(\mathcal{K})$$

$\Leftrightarrow$ prediction cannot influence observation and knowledge

$$^{\mathsf{I}}\mathsf{S}(\hat{\Delta}|\Delta, \mathcal{K}) = \mathsf{S}(\hat{\Delta}|\mathcal{K})$$

$\Leftrightarrow$ the decision rule is left to the fate↰: no general requirements apply.

Optimal Predictor in FPD Sense

Proposition 32 (Optimal Prediction)

Under (45), (178), (179) and (181), the optimal predictor is deterministic generating the optimal prediction

$$
\begin{aligned}
{}^{o}\hat{\Delta} &\in \mathrm{Arg}\min_{\hat{\Delta}\in\Delta^\star} \int_{X_1^\star, X_0^\star} \mathsf{f}(X_1, X_0|\mathcal{K}) \ln\left(\frac{\mathsf{f}(X_1, X_0|\mathcal{K})}{\mathsf{f}(X_1, X_0|\hat{\Delta}, \mathcal{K})}\right) \mathrm{d}X_1\, \mathrm{d}X_0 \quad (182)\\
&= \mathrm{Arg}\min_{\hat{\Delta}\in\Delta^\star} \int_{X_1^\star, X_0^\star} \mathsf{f}(X_1, X_0|\mathcal{K}) \ln\left(\frac{\mathsf{f}(\hat{\Delta}|\mathcal{K})}{\mathsf{f}(\Delta = \hat{\Delta}|X_1, \mathcal{K})}\right) \mathrm{d}X_1\, \mathrm{d}X_0
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{f}(X_1, X_0|\mathcal{K}) &\propto \mathsf{f}(X_1|X_0, \mathcal{K})\mathsf{f}(X_0|\mathcal{K})\mathsf{f}(\mathcal{K})\\
\mathsf{f}(X_1, X_0|\hat{\Delta}, \mathcal{K}) &\propto \mathsf{f}(\Delta = \hat{\Delta}|X_1, \mathcal{K})\mathsf{f}(X_1, X_0|\mathcal{K})\\
\mathsf{f}(\hat{\Delta}|\mathcal{K}) &= \int_{X_1^\star, X_0^\star} \mathsf{f}(X_1, X_0|\mathcal{K})\mathsf{f}(\Delta = \hat{\Delta}|X_1, \mathcal{K})\, \mathrm{d}X_1\, \mathrm{d}X_0.
\end{aligned}
$$

Proof By a direct use of basic relations between pds. $\square$

Remarks on One-Step-Ahead Prediction

- The result can be interpreted as approximation of predictive pd↰ by the observation model↰.
- The standard Bayesian prediction, determined by performance index $\mathsf{I}(\Delta, \hat{\Delta}, \mathcal{K})$, can be formally cast into the FPD by defining the ideal pd $^{\mathsf{I}}\mathsf{f}(\Delta, X_1, X_0, \hat{\Delta}, \mathcal{K}) \propto \mathsf{f}(\Delta|X_1, \mathcal{K})\mathsf{f}(X_1|X_0, \mathcal{K}) \exp[-\mathsf{I}(\Delta, \hat{\Delta}, \mathcal{K})/\lambda]$, $\lambda > 0$, $\lambda \approx 0$, cf. (81).
- The significant role of the system model↰ in (186), whose output is predicted, is unusual and highly plausible. Standard Bayesian DM selects performance index↰ $\mathsf{I}(\Delta, \hat{\Delta}, \mathcal{K})$ unrelated to the parametric model, often, as squared norm of the difference $\Delta - \hat{\Delta}$. This is a good choice for system models close to linear-Gaussian ones. It can be rather bad choice for heavy-tailed models.
- Other actions, like system inputs, may influence the prediction knowledge. Then, $\mathcal{K}_{\hat{\Delta}} \equiv (\mathcal{K}_U, U)$ and inputs influence both value of the observation and its prediction.

Static Design with Unknown Parameter

The static design with time invariant hidden $X = X_0 = \Theta$ needs only

- behaviour $B = (\mathcal{G}_A, A, \mathcal{K}_A) =$(ignorance,action,knowledge) =((parameter,unmade observations),action,knowledge) $= ((\Theta, \Delta = \Delta_1), A = A_1, \mathcal{K}_A = \mathcal{K}_0 = \mathcal{K})$
- admissible decision rules meeting (45) $\mathsf{S}(A|\Theta, \mathcal{K}) = \mathsf{S}(A|\mathcal{K})$
- parametric model $\mathsf{f}(\Delta|\Theta, A, \mathcal{K})$ & its ideal ${}^{\mathsf{I}}\mathsf{f}(\Delta|\Theta, A, \mathcal{K})$
- prior pd $\mathsf{f}(\Theta|\mathcal{K})\mathsf{f}(\mathcal{K})$ & its ideal ${}^{\mathsf{I}}\mathsf{f}(\Theta|\mathcal{K})\,{}^{\mathsf{I}}\mathsf{f}(\mathcal{K})$
- ideal decision rule ${}^{\mathsf{I}}\mathsf{S}(A|\Theta, \mathcal{K})$.

The general FPD, Proposition 25, provides the optimal decision rule

$$
{}^{O}\mathsf{S}(A|\mathcal{K}) \propto {}^{\mathsf{I}}\mathsf{S}(A|\mathcal{K}) \exp[-\omega(A, \mathcal{K})] \tag{183}
$$

$$
\ln({}^{\mathsf{I}}\mathsf{S}(A|\mathcal{K})) = \int_{\Theta^\star} \ln({}^{\mathsf{I}}\mathsf{S}(A|\Theta, \mathcal{K}))\,{}^{\mathsf{I}}\mathsf{f}(\Theta|\mathcal{K})\,\mathrm{d}\Theta, \quad \omega(A, \mathcal{K})
$$

$$
= \int_{\Theta^\star} \mathsf{f}(\Theta|\mathcal{K}) \int_{\Delta^\star} \mathsf{f}(\Delta|\Theta, A, \mathcal{K}) \ln\left(\frac{\mathsf{f}(\Delta|\Theta, A, \mathcal{K})}{{}^{\mathsf{I}}\mathsf{f}(\Delta|\Theta, A, \mathcal{K})}\right) \mathrm{d}\Delta\,\mathrm{d}\Theta
$$

Example 13 (Point Estimation)

aim	*to estimate unknown parameter* $\Theta \in \Theta^\star$ *entering parametric model uninfluenced by the estimate chosen* $$f(\Delta\|\Theta, \mathcal{K}) = f(\Delta\|\Theta, \hat{\Theta}, \mathcal{K}) \quad (184)$$
system	*modelled World part*
action	$\hat{\Theta} \in \Theta_\star \subset \Theta^\star$
knowledge	$\mathcal{K}$ *entering parametric model and prior pd* $f(\Theta\|\mathcal{K})$
ignorance	*estimated parameter* Θ *and observation* Δ
uncertainty	*anything preventing to determine fully* Θ *from* $\mathcal{K}$
constraint	$\Theta_\star$, *computational complexity*
dynamics	*none*

Point Estimation as Static Design with Unknown Θ

DM elements↰ relevant to the point estimation are

- behaviour↰ $B=(\mathcal{G}_A, A, \mathcal{K}_A)=$(ignorance↰,action↰,knowledge↰) =((unmade observations,parameter),estimate,knowledge) $= ((\Delta = \Delta_1, \Theta), \hat{\Theta}, \mathcal{K}_{\hat{\Theta}} = \mathcal{K})$
- admissible decision rule↰s meeting (45) $\mathsf{S}(\hat{\Theta}|\Theta, \mathcal{K}) = \mathsf{S}(\hat{\Theta}|\mathcal{K})$
- parametric model↰ $\mathsf{f}(\Delta|\Theta, \hat{\Theta}, \mathcal{K}) = \mathsf{f}(\Delta|\Theta, \mathcal{K})$ & its ideal ${}^{\mathsf{I}}\mathsf{f}(\Delta|\Theta, \hat{\Theta}, \mathcal{K})$
- prior pd↰ $\mathsf{f}(\Theta|\mathcal{K})\mathsf{f}(\mathcal{K})$ & its ideal ${}^{\mathsf{I}}\mathsf{f}(\Theta|\mathcal{K})\,{}^{\mathsf{I}}\mathsf{f}(\mathcal{K})$
- ideal decision rule↰ ${}^{\mathsf{I}}\mathsf{S}(\hat{\Theta}|\Theta, \mathcal{K})$.

The following options of red elements to be specified seem to be "natural"

$$\begin{aligned}
&{}^{\mathsf{I}}\mathsf{f}(\Theta|\mathcal{K}) = \mathsf{f}(\Theta|\mathcal{K}) \Leftrightarrow \text{ DM preserves the relation of } \Theta \ \& \ \mathcal{K} \\
&{}^{\mathsf{I}}\mathsf{f}(\Delta|\Theta, \hat{\Theta}, \mathcal{K}) = \mathsf{f}(\Delta|\hat{\Theta}, \mathcal{K}) \Leftrightarrow \text{ ideally the estimate relates observation} \\
&\text{to knowledge exactly as the (unknown) parameter} \\
&{}^{\mathsf{I}}\mathsf{S}(\hat{\Theta}|\Theta, \mathcal{K}) = \mathsf{S}(\hat{\Theta}|\mathcal{K}) \Leftrightarrow \text{ leave to the fate↰ option expresses lack of} \\
&\text{wishes on the designed decision rule↰.} \qquad (185)
\end{aligned}$$

Optimal Point Estimator in FPD Sense

Proposition 33 (Optimal Point Estimator)

The static FPD↰ determined by options (185) provides the deterministic point estimator generating the optimal estimate as minimiser of the KLD↰ of the predictive pd↰ $\mathsf{f}(\Delta|\mathcal{K}) = \int_{\Theta^\star} \mathsf{f}(\Delta|\Theta,\mathcal{K})$ *on parametric model↰ with the estimate* $\hat{\Theta}$ *"plug-in" instead of the unknown parameter* Θ

$$^{O}\hat{\Theta} \in \mathrm{Arg}\min_{\hat{\Theta}\in\Theta_\star} \mathsf{D}(\mathsf{f}(\Delta|\mathcal{K})||\mathsf{f}(\Delta|\hat{\Theta},\mathcal{K})). \quad (186)$$

Proof Due to the leave to the fate option, the optimised KLD is linear in the optimised decision rule. Direct evaluations respecting (184), natural conditions of DM↰ (45) and with DM elements (185) show that $\mathsf{D}(\mathsf{f}_\mathsf{S}(B)||\,^{\mathsf{I}}\mathsf{f}) = \int_{\hat{\Theta}^\star=\Theta_\star} \mathsf{S}(\hat{\Theta}|\mathcal{K}) \int_{\Delta^\star} \mathsf{f}(\Delta|\mathcal{K}) \ln\left(\frac{1}{\mathsf{f}(\Delta|\hat{\Theta},\mathcal{K})}\right) \mathrm{d}\Delta\, \mathrm{d}\hat{\Theta} +$ term independent of $\mathsf{S}(\hat{\Theta}|\mathcal{K})$. Thus, the optimal rule is to concentrate on minimiser of the Kerridge inaccuracy, which coincides with minimiser of the KLD. □

Remarks on Point Estimation

- The result can be interpreted as approximation of predictive pd↰ by the predictor obtained by plug-in point estimate $\hat{\Theta}$ into the parametric model↰, cf. (85).
- Recall that the standard Bayesian estimation, determined by performance index $\mathsf{I}(\Theta, \hat{\Theta}, \mathcal{K})$, can be formally cast into the FPD by defining the ideal pd ${}^{\mathsf{I}}\mathsf{f}(\Delta, \Theta, \hat{\Theta}, \mathcal{K}) \propto \mathsf{f}(\Delta|\Theta, \mathcal{K}) \exp[-\mathsf{I}(\Theta, \hat{\Theta}, \mathcal{K})/\lambda]$, $\lambda > 0$, $\lambda \approx 0$, cf. (81).
- The significant role of the parametric model↰ in (186), whose parameter is estimated, is unusual and highly plausible. Standard Bayesian DM selects performance index↰ $\mathsf{I}(\Theta, \hat{\Theta}, \mathcal{K})$ unrelated to the parametric model, often, as squared norm of the difference $\Theta - \hat{\Theta}$. This is good choice for parametric models close to linear-Gaussian ones. It can be rather bad choice for non-symmetric and/or heavy-tailed models.

Example 14 (Testing of Hypotheses)

aim	*to estimate the pointer* $\vartheta \in \vartheta^\star \equiv \{1, \ldots, \|\vartheta^\star\|\}$, $\|\vartheta^\star\| < \infty$ *to the hypothesis* H_θ *about system model* $$\mathsf{f}(\Delta\|H_\vartheta, \mathcal{K}) = \mathsf{f}(\Delta\|H_\vartheta, \hat{\vartheta}, \mathcal{K}) \quad (187)$$ *by* $\hat{\vartheta} \in \vartheta_\star \subset \vartheta^\star$
system	*modelled World part*
action	$\hat{\vartheta} \in \vartheta_\star \subset \vartheta^\star$
knowledge	$\mathcal{K}$ *entering the system model and prior pd* $\mathsf{f}(\vartheta\|\mathcal{K})$
ignorance	*estimated parameter* ϑ *and observation* Δ
uncertainty	*anything preventing to determine fully* ϑ *from* $\mathcal{K}$
constraint	$\vartheta_\star$, *computational complexity implied by the excessive* $\|\vartheta^\star\|$
dynamics	*none*

Hypotheses Testing as Static Design with Unknown ϑ

DM elements↰ relevant to the point estimation are

- behaviour↰ $B=(\mathcal{G}_A,A,\mathcal{K}_A)=$(ignorance↰,action↰,knowledge↰) =((unmade observations,pointer to the most plausible hypothesis),pointer estimate,knowledge) $= ((\Delta = \Delta_1, \vartheta), \hat{\vartheta}, \mathcal{K}_{\hat{\vartheta}} = \mathcal{K})$
- admissible decision rule↰s meeting (45) $\mathsf{S}(\hat{\vartheta}|\vartheta,\mathcal{K}) = \mathsf{S}(\hat{\vartheta}|\mathcal{K})$
- system model↰ $\mathsf{f}(\Delta|H_\vartheta,\hat{\vartheta},\mathcal{K}) = \mathsf{f}(\Delta|H_\vartheta,\mathcal{K})$ & its ideal ${}^{\mathsf{I}}\mathsf{f}(\Delta|H_\vartheta,\hat{\vartheta},\mathcal{K})$
- prior pd↰ $\mathsf{f}(\vartheta|\mathcal{K})\mathsf{f}(\mathcal{K})$ & its ideal ${}^{\mathsf{I}}\mathsf{f}(\vartheta|\mathcal{K})\,{}^{\mathsf{I}}\mathsf{f}(\mathcal{K})$
- ideal decision rule↰ ${}^{\mathsf{I}}\mathsf{S}(\hat{\vartheta}|\vartheta,\mathcal{K})$.

Observe: hypotheses testing coincides with parameter estimation with discrete-valued parameter $\vartheta \leftrightarrow \Theta$. Thus, we can focus on specificity of hypotheses testing.

Remarks on Hypotheses Testing I

- The system model $f(\Delta|H_\vartheta, \mathcal{K})$ is model parameterised by $\vartheta \in \vartheta^\star$.
- Unlike in classical hypotheses testing [Rao87a], the testing is performed within a completely specified set of alternatives.
- The hypotheses testing is usually performed with the knowledge (gradually) extended by by data, say d^t. Bayesian estimation, Proposition 15, provides the key posterior pd $f(\vartheta|d^t, \mathcal{K})$. Discrete nature of ϑ implies that this pd quickly concentrates on a small subset of $\vartheta_\star$ containing often single point, see Proposition 18. Thus, with any reasonable ideal pd, single hypothesis within $\vartheta_\star$ is accepted as the most plausible one even when none of them is correct.
- The needed pds $\{f(\Delta|H_\vartheta, \mathcal{K})\}_{\vartheta \in \vartheta^\star}$ are rarely obtained directly. Instead, they are predictive pds obtained through filtering or parameter estimation and prediction, Propositions 14, 18. Influence of prior pds within these "auxiliary" tasks on values of $f(\vartheta|d^t, \mathcal{K})$ is quite significant.

Remarks on Hypotheses Testing II

- Other actions, like control, are usually present and usually meet natural conditions of DM↰ (45). Then the decision rules generating them cancel in the formula for $f(\vartheta|d^t, \mathcal{K})$.
- The testing of hypotheses is extremely powerful technique in spite of its formal simplicity. It is especially true when dealing with the predictive pd↰s evaluated for each hypothesis by filtering or estimation. Non-Bayesian treatment of such compound hypotheses, [Rao87a], is far from being trivial. The Bayesian solution brought a whole set of novel and efficient solutions of so called
- *structure estimation* , which selects among alternative parametric models differing in functional form, order of regression vector↰ or selection of significant variables to be used in the system model↰, [K83, KK88, Ber98].

Proposition 34 (Structure Estimation)

Let parametric model↰s $\{\mathsf{f}(\Delta_t|\Theta_\vartheta, A_t, \mathcal{K}_{t-1}, \vartheta)\}_{\vartheta\in\vartheta^\star;t\in t^\star}$ *be candidates for describing of a system. Let the respective unknown parameter* Θ_ϑ *be described by a prior pd↰* $\mathsf{f}(\Theta_\vartheta|\vartheta, \mathcal{K}_0)$ *and prior pd↰* $\mathsf{f}(\vartheta|\mathcal{K}_0)$ *be prior probabilities of hypotheses. Let the possible additional action↰s* A_t *(like system inputs) meet natural conditions of DM↰. Then, posterior pd on hypotheses, needed for hypotheses testing, are*

$$\begin{aligned} \mathsf{f}(\vartheta|\mathcal{K}_t) &\propto \frac{\mathsf{J}_\vartheta(Kt)}{\mathsf{J}_\vartheta(\mathcal{K}_{t-1})}\mathsf{f}(\vartheta|\mathcal{K}_{t-1}) \\ \mathsf{J}_\vartheta(\mathcal{K}_t) &= \int_{\Theta_\vartheta^\star}\prod_{\tau=1}^{t}\mathsf{f}(\Delta_\tau|\Theta_\vartheta, A_\tau, \mathcal{K}_{\tau-1})\mathsf{f}(\Theta_\vartheta|\mathcal{K}_0)\,\mathrm{d}\Theta_\vartheta. \end{aligned} \tag{188}$$

Proof It uses basic algebra with pds and represents a version of Proposition 15, respecting natural conditions of DM↰ adopted for all involved decisions.

Remarks on Structure Estimation

- Structure estimation can be formulated and solved in conjunction with filtering.
- The knowledge can generally depend on the structure of the model within which it is used.
- Mechanical ways of generating list of hypotheses make $|\vartheta^\star|$ extremely large and consequently the their testing infeasible.
- Hypotheses are usually created gradually. It opens a question, how to extend the existing set of hypotheses and how to exploit former data so that the new hypothesis is compared in a fair way. A lot of partial steps have been done in this respect but a systematic design and analysis are missing.

Static Design with No Hidden Quantity

The static design without hidden quantities needs only

- behaviour↰ B=($\mathcal{G}_A$,A,$\mathcal{K}_A$)=(ignorance↰,action↰,knowledge↰) =(unmade observations,action,knowledge) $= (\Delta = \Delta_1, A = A_1, \mathcal{K}_A = \mathcal{K}_0 = \mathcal{K})$
- admissible decision rule↰s $\mathsf{f}(A|\mathcal{K}) = \mathsf{f}(A|\mathcal{K})$
- system model↰ $\mathsf{f}(\Delta|A,\mathcal{K})$ & its ideal $^{\mathsf{I}}\mathsf{f}(\Delta|A,\mathcal{K})$
- ideal decision rule↰ $^{\mathsf{I}}\mathsf{f}(A|\mathcal{K})$.

The general FPD↰, Proposition 25, provides the optimal decision rule↰

$$\begin{aligned} {}^{O}\mathsf{f}(A|\mathcal{K}) &\propto {}^{\mathsf{I}}\mathsf{f}(A|\mathcal{K})\exp[-\omega(A,\mathcal{K})] \qquad (189)\\ \omega(A,\mathcal{K}) &= \int_{\Delta^\star} \mathsf{f}(\Delta|A,\mathcal{K}) \ln\left(\frac{\mathsf{f}(\Delta|A,\mathcal{K})}{{}^{\mathsf{I}}\mathsf{f}(\Delta|A,\mathcal{K})}\right) \mathrm{d}\Delta. \end{aligned}$$

One-Step-Ahead Control

Example 15 (One-Step-Ahead Control)

aim	*to select system input* $U \in U^\star$ *entering system modelled by* $\mathrm{f}(\Delta \vert u, \mathcal{K})$ *so that the observation* Δ *is close to a set point* ${}^s\Delta \in \Delta^\star$
system	*modelled World part to be influenced*
action	$u \in u^\star$
knowledge	$\mathcal{K}$ *entering system model and prior pd together with set poi*
ignorance	*the unmade observation* Δ
uncertainty	*anything preventing to determine fully* Δ *from* u *and* $\mathcal{K}$
constraint	$u^\star$, *computational complexity*
dynamics	*none*

One-Step-Ahead Control as Static Design

DM elements↰ relevant to one-step-ahead control are

- behaviour↰ B=($\mathcal{G}_A$,A,$\mathcal{K}_A$)=(ignorance↰,action↰,knowledge↰) =(unmade observations,parameter,input,knowledge) = $(\Delta, u, \mathcal{K}_{u^\star} = \mathcal{K})$
- admissible decision rule↰s (control laws) $\mathsf{S}(u|\mathcal{K})$
- system model↰ $\mathsf{f}(\Delta|u,\mathcal{K})$ & its ideal $^{\mathsf{I}}\mathsf{f}(\Delta|u,\mathcal{K})$
- prior pd↰ $\mathsf{f}(\mathcal{K})$ & its ideal $^{\mathsf{I}}\mathsf{f}(\mathcal{K})$
- ideal decision rule↰ $^{\mathsf{I}}\mathsf{S}(u|\mathcal{K})$.

The following options of red elements to be specified seem to be "natural"

$$^{\mathsf{I}}\mathsf{f}(\mathcal{K}) = \mathsf{f}(\mathcal{K}) \Leftrightarrow \text{ input } U \text{ has no influence on knowledge } \mathcal{K}$$
$$^{\mathsf{I}}\mathsf{f}(\Delta|U,\mathcal{K}) = \mathsf{f}(\Delta|\,^{s}U,\mathcal{K}) \text{ with } \mathsf{f}(\,^{s}\Delta|\,^{s}U,\mathcal{K}) \geq \mathsf{f}(\Delta|U,\mathcal{K}) \text{ on } (\Delta^\star, U^\star)$$
$$^{\mathsf{I}}\mathsf{S}(U|\mathcal{K}) \text{ a pdf with support in } U^\star.$$

The solution (189) is directly applicable with correspondence $U = A$.

Remarks on One-Step-Ahead Control

- The input U influences directly the system↰ and thus observation↰.
- The standard one-step-ahead control, determined by performance index $\mathsf{I}(\Delta, U, \mathcal{K})$, casts into the FPD by using the ideal pd ${}^{\mathsf{I}}\mathsf{f}(\Delta, U, \mathcal{K}) \propto \mathsf{f}(\Delta|U, \mathcal{K}) \exp[-\mathsf{I}(\Delta, U, \mathcal{K})/\lambda]$, $\lambda > 0$, $\lambda \approx 0$, cf. (81).
- Due to uncertainty↰ no input (including ${}^{s}U$ whose existence is assumed) can enforce Δ to coincide with the set point ${}^{s}\Delta$: the corresponding ideal (called the most optimistic one [**?**]) is non-degenerated pd↰. It "penalises" the deviations Δ and ${}^{s}\Delta$ in harmony with the system model respecting the character of uncertainty. The popular quadratic performance indices [AM89] are obtained for Gaussian system models. If the system model is far from Gaussian case, their use is doubtful.
- The danger of solving dynamic control via chaining of one-step-ahead control can hardly be over-stressed [KHB+85]. Thus, it should be considered in really static cases or in suboptimal strategies.

The following tasks focus on DM problems in which their *dynamic* character plays a substantial role.

Sequential Estimation

- *sequential estimation* decides, whether to process a new observations or whether to stop; when stopping it provides a final estimate of an unknown parameter.
- Sequential estimation balances non-negligible costs connected with acquiring observation with costs induced by imprecisions of the final estimate.
- Note that
  - Unlike majority decision tasks with a fixed horizon $h < \infty$, the sequential estimation deals with a potentially infinite h.
  - Testing of hypotheses is a specific case of estimation. Thus, the subsequent treatment can be applied to it, too.
  - Sequential estimation was at roots of the theory of statistical decision functions, we build on [Wal50].
- The problem of sequential estimation is formulated in standard Bayesian way. The generalisation to FPD has not been inspected yet.

Formalisation of Sequential Estimation

Sequential point estimation can be cast in our framework as follows.

- $B \equiv [\mathcal{G}_t \equiv (\Theta, \Delta_t), A_t \equiv (\hat{\Theta}_t, s_t), \mathcal{K}_{t-1}]$
  [(unknown parameter,observations),(estimate,stopping flag),data at disposal].
- Admissible strategies consist of rules $\mathsf{S}_t : \mathcal{K}_{t-1}{}^\star \to (\hat{\Theta}_t^\star, s_t{}^\star)$, $\Theta^\star \subset \hat{\Theta}^\star$, $s_t{}^\star \equiv$ {stop measuring and estimate Θ, make a new observation} $\equiv \{0, 1\}$,
- Loss

$$Z = \begin{cases} \sum_{\tau \leq t} c(\mathcal{K}_{\tau-1}) + z(\Theta, \hat{\Theta}_t, \mathcal{K}_{t-1}) & \text{if} \quad s_t = 0 \ \& \ s_\tau = 1, \ \forall \tau < t \\ \sum_{\tau \leq t} c(\mathcal{K}_{\tau-1}) & \text{if} \quad s_\tau = 1 \ \forall \tau \leq t \end{cases} \tag{190}$$

where $z(\Theta, \hat{\Theta}_t, \mathcal{K}_{t-1})$ measures a distance of Θ and its estimate $\hat{\Theta}$. $c(\mathcal{K}_{\tau-1})$ denotes a positive price of τth observation.

Optimal Sequential Estimation I

Proposition 35 (Sequential Estimation)

Let us consider the sequential estimation and assume that there is an admissible strategy for which the expected loss is finite. The following inequalities express the sufficient condition for an index t to be the time moment at which observation should be stopped

$$\mathsf{E}\left[\left(z(\Theta, \hat{\Theta}_t, \mathcal{K}_{t-1}) - z(\Theta, \hat{\Theta}_{t+k}, \mathcal{K}_{t+k-1}) - \sum_{\tau>t}^{t+k} c(\mathcal{K}_{\tau-1})\right) |\mathcal{K}_{t-1}\right] \leq 0$$
$$\forall k = 1, 2, \ldots. \tag{191}$$

In (191), $\hat{\Theta}_{t+k}, \ k = 0, 1, 2, \ldots$ denote parameter estimate based on $\mathcal{K}_{t+k-1}$ minimising $\mathsf{E}[z(\Theta, \hat{\Theta}, \mathcal{K}_{t+k-1})]$.

Proof Let (191) be fulfilled. Then, combining the form of the loss (190), the fact that the optimal stopping time has to be determined using

Optimal Sequential Estimation II

its knowledge and finiteness of the loss for the optimal solution we get, $\forall k = 1, 2, \ldots,$

$$\begin{aligned} & \mathsf{E}\left[\left(z(\Theta, \hat{\Theta}_t, \mathcal{K}_{t-1}) + \sum_{\tau=1}^{t} c(\mathcal{K}_{\tau-1})\right) | \mathcal{K}_{t-1}\right] \\ \leq \; & \mathsf{E}\left[\left(z(\Theta, \hat{\Theta}_{t+k}, \mathcal{K}_{t+k-1}) + \sum_{\tau=1}^{t+k} c(\mathcal{K}_{\tau-1})\right) | \mathcal{K}_{t-1}\right]. \end{aligned}$$

Using isotonicity of the expectation (taken over $\mathcal{K}_{t-1}$), we find that the chosen decision cannot be improved by any estimate that uses more observations than the inspected one. $\square$

Remarks on Sequential Estimation

- The implementation of the proposed decision rule requires the generalised Bayesian estimate (posterior pd) given in Proposition 15.
- The ability to evaluate $\mathsf{E}\left[z(\Theta, \hat{\Theta}_{t+k}, \mathcal{K}_{t+k-1})|\mathcal{K}_{t-1}\right]$ is decisive for a practical solvability of the problem.
- Stopping rules used for speeding up extensive simulations [RK98] based on a simple sequential estimation serve as an example of their, still underestimated, usefulness.
- The dependence of the observation price on available knowledge can be effectively exploited when the sequential estimation is performed in an inner loop of some optimisation process: the closer we are to the optimum the lower this price can be. This fact was used, for instance, in [KH94].

Multi-Step-Ahead Prediction

Multi-step-ahead prediction extrapolates known data to a more distant future. This tasks extends one-step-ahead prediction, Example 12, and fits the considered traditional Bayesian DM as follows.

- $B \equiv (\Delta_{t+j}, \hat{\Delta}_{t+j|t-1}, \mathcal{K}_{t-1}) \equiv$ (future observations at time $t+j$, $j \geq 1$,prediction of observations at time $t+j$,knowledge at disposal at time $t-1$).
- Admissible decision rules $\mathsf{S}_t : \mathcal{K}_{t-1}{}^{\star} \rightarrow \hat{\Delta}^{\star}_{t+j|t-1}$.
- Loss $\mathsf{Z}(\Delta_{t+j}, \hat{\Delta}_{t+j|t-1}, \mathcal{K}_{t-1})$ measures $\mathcal{K}_t$-dependent distance of Δ_{t+j} and $\hat{\Delta}_{t+j|t-1}$.
- The basic DM lemma, Proposition 10, the optimal decision rule is deterministic and generates the optimal point prediction

$$\hat{\Delta}_{t+j|t-1} \in \mathrm{Arg}\min_{\hat{\Delta}\in\hat{\Delta}^{\star}} \int \mathsf{Z}(\Delta_{t+j}, \hat{\Delta}, \mathcal{K}_{t-1},)\mathsf{f}(\Delta_{t+j}|\mathcal{K}_{t-1})\,\mathrm{d}\Delta_{t+j}, \quad (192)$$

  where the predicted Δ_{t+j} is assumed independent of the prediction.
- The evaluation of the optimal prediction $\hat{\Delta}_{t+j|t-1}$ (192) requires the multi-step-ahead predictive pd $\mathsf{f}(\Delta_{t+j}|\mathcal{K}_{t-1})$.

Multi-Step-Ahead Predictive Pd

The multi-step-ahead predictive pd is usually constructed from predictive pd, which may generally depend on other action. For specificity, let the considered the observation is influenced by system input $U_t \in U_t{}^\star$ generated by a randomised control strategy $\mathsf{f}(U_t|\mathcal{K}_{t-1})$, $t \in t^\star$, i.e. the predictive pd has the form $\mathsf{f}(\Delta_t|U_t,\mathcal{K}_{t-1})$, $t \in t^\star$.
The basic rules for pds, Proposition 5, imply

$$\mathsf{f}(\Delta_{t+j}|\mathcal{K}_{t-1}) = \int \mathsf{f}(\Delta^{t:t+j}, U^{t:t+j}|\mathcal{K}_{t-1})\,\mathrm{d}\big(\Delta^{t:t+j-1}, U^{t:t+j}\big)$$

$$= \int \prod_{\tau=t}^{t+j} \mathsf{f}(\Delta_\tau|U_\tau,\mathcal{K}_{\tau-1})\mathsf{f}(u_\tau|\mathcal{K}_{\tau-1})\,\mathrm{d}\big(\Delta^{t:t+j-1}, U^{t:t+j}\big). \qquad (193)$$

Remarks on Multi-Step-Ahead Prediction I

- The need to know the control strategy makes the main difference of this task from one-step-ahead prediction where just knowledge of input values U_t is sufficient.
- The computed marginal pd $\mathsf{f}(\Delta_{t+j}|U_t, \mathcal{K}_{t-1}),\ j > 1$ is in generic case much flatter than the one-step-ahead predictor. It is seen from the fact that it is obtained by integration (averaging) over intermediate predicted values.
  This corresponds with common experience that it is much harder (less reliable) to make a long term prediction. The uncertainty quickly increases with increasing j.
- The integrations over the intermediate quantities is done also over their values in conditioning. This makes multi-step-ahead prediction a highly non-linear task.

- Sometimes, the parametric model is directly chosen to have the gap $j \geq 1$

$$f(\Delta_{t+j}|\Theta, U_{t+j}, \mathcal{K}_{t+j-1}) = f(\Delta_{t+j}|\Theta, U_t, \mathcal{K}_{t-1}). \qquad (194)$$

  Then, there is no computational and formal difference from the one-step-ahead prediction. The parametric model and consequently the prediction quality is of course worse as the assumption (194) rarely reflects reality.

- The predictors of the type (194) are used in connection with adaptive controllers called MUSMAR [MCG93].

Filtering Derivatives

- Filtering, Proposition 14, provides pds $\mathsf{f}(X_t|\mathcal{K}_t)$ and $\mathsf{f}(X_t|A_t, \mathcal{K}_{t-1})$. Using basic rules for pds, Proposition 5, it is formally simple to obtain multi-step-ahead predictors $\mathsf{f}(X_{t+j}|A_t, \mathcal{K}_{t-1})$ similarly as (193).
- The fact that we never observe directly time varying X_t calls for a novel task called smoothing.
- *smoothing* is evaluation of the pd $\mathsf{f}(X_{t-j}|\mathcal{K}_t)$ $j > 1.$ using also the measured data reflecting newer quantities
  Its construction is cast in our framework as follows.
- $B \equiv (X_{t-j}, \hat{X}_{t-j|t}, \mathcal{K}_t) \equiv$
  (unknown hidden quantity at time $t-j$, $j \geq 1$, smoothed estimate of X_{t-j} based on data,data at disposal at time t).
- Admissible decision rules are of the form $\mathsf{S} : \mathcal{K}_t{}^{\star} \to \hat{X}^{\star}_{t-j|t}$.
- Loss Z measures $\mathcal{K}_t$-depended distance of X_{t-j} and $\hat{X}_{t-j|t}$.

Optimal Smoothing

The system is assumed to have input the input↰ U_t (meeting natural conditions of DM↰, (45)), the output↰ Y_t and the state X_t. It is modelled by the observation model↰ and state evolution model↰. The basic DM lemma, Proposition 10, provides the optimal estimate

$$\hat{X}_{t-j|t} \in \mathrm{Arg} \min_{\hat{X} \in \hat{X}^{\star}_{t-j}} \int \mathsf{Z}(X_{t-j}, \hat{X}_{t-j}, \mathcal{K}_t) \mathsf{f}(X_{t-j}|\mathcal{K}_t) \, \mathrm{d}X_{t-j}. \tag{195}$$

The calculus with pds, Proposition 5 and natural conditions of DM↰ (45) provide the pd $\mathsf{f}(X_{t-j}|\mathcal{K}_t)$ needed in (195):

$$\begin{aligned} & \mathsf{f}(X_{t-j}|\mathcal{K}_t) \propto \mathsf{f}(X_{t-j}, Y^{t-j+1:t}, U^{t-j+1:t}|\mathcal{K}_{t-j}) \\ \propto \quad & \mathsf{f}(X_{t-j}|\mathcal{K}_{t-j}) \int \prod_{\tau=t-j+1}^{t} \mathsf{f}(Y_\tau|U_\tau, X_\tau) \mathsf{f}(X_\tau|U_t, X_{\tau-1}) \, \mathrm{d}X^{t-j+1:t}. \end{aligned}$$

The integrand consists of available models with known data inserted.

Remarks on Smoothing

- The result is product of the pd $f(X_{t-j}|\mathcal{K}_{t-j})$, gained by filtering↰, Proposition 14, and the integration result depending on X_{t-j}.
- The filtering represents the main computational burden with smoothing↰.
- Generically, the pdf describing smoother is more narrow than the filtering result. It corresponds with intuition that a good retrospective estimate of unobserved state based on a wider knowledge is more precise than a good immediate estimate.

Multi-Step-Ahead Control and Its Types

Multi-step-ahead controls is the most general dynamic DM task in which actions are inputs influencing behaviour including some hidden quantity. Unlike one-stage-ahead control, Example 12, multi-step-ahead control considers horizon $h > 1$ and (control) strategy pushes system outputs Y_t and states X_t to set points ${}^sY_t$ and ${}^sX_t$ while keeping inputs U_t close to their reference (set point) ${}^sU_t$ for $t \in t^\star = \{1, \ldots, h\}$. Differences in available knowledge about set points leads to different control problems.

- *tracking problem* is characterised by uncertain set points, which are not fully known in beforehand and have to be measured, i.e. observation $\Delta_t = (Y_t, {}^sY_t, {}^sU_t, {}^sX_t)$ and modelled by observation model.
- *tracking problem with pre-programming* arises whenever some future set points, say ${}^sY_t$ are known beforehand, i.e. knowledge $\mathcal{K}_t$ contains ${}^sY_\tau,\ \tau > t$. Thus, modelling of this set point is superfluous.
- *regulation problem* is characterised by known constant set points.

Multi-Step-Ahead Control, Ist Formalisation & Solution

This tracking problem↰ is formalised as follows.

- $B \equiv [\mathcal{G}_t, U_t, \mathcal{K}_{t-1}]$ =[ignorance↰,action↰,knowledge↰]
  =[(future observations,states),inputs, observations], $t \leq h$.
- Admissible strategy consists of sequence rules (control laws)
  $\{\mathsf{S}_t : {\mathcal{K}_{t-1}}^\star \rightarrow {u_t}^\star\}_{t \in t^\star}$.
- Knowledge evolves $\mathcal{K}_t = (K_{t-1}, \Delta_t, U_t)$ starting from prior one $\mathcal{K}_0$ and is quantified by prior pd↰ $\mathsf{f}(X_0)$.
- Loss Z measures distance of B to set points ${}^sY^h$, ${}^sU^h$, ${}^sX^h$.
- The observation model↰ $\mathsf{f}(Y_t, {}^sY_t, {}^sU_t, {}^sX_t | U_t, X_t, \mathcal{K}_{t-1})$ and the time evolution model↰ are DM elements↰ needed.

The optimal strategy is described by dynamic programming Proposition 11 exploiting stochastic filtering, Proposition pro:P9↰. If an ideal pd↰ is specified instead of the loss, the optimal strategy in FPD↰ sense is described by Proposition 25.

Remarks on Multi-Step-Ahead Control I

- The described general cases covers the situation when we try to follow evolution of another uncertain object. Rescue/military interpretations of this case are straightforward. The key message is that the dynamics of the uncertain target has to be modelled for finding the optimal strategy.
  The situation is simplified whenever target values are (partially) known. Then the corresponding outer models reduce formally to Dirac delta functions on a given known support.
- The modelling of set points is often neglected and their future time-invariance implicitly assumed. It corresponds with their modelling by random walk with time-varying dispersion [Pet84]. Use of such approximation in a pre-programming problem leads to worse-than-possible controller.
- The optimal controller exploits results of *generalised* Bayesian filtering: no point estimate of hidden variables has to be selected.

Remarks on Multi-Step-Ahead Control II

- The special case with unknown time invariant parameters $\Theta = X_t = X_{t-1}$ is the central topic of model-based adaptive control [AW89].
- Even adaptive control, which uses parameter estimation, Proposition 15, instead of filtering, suffers generally from computational complexity (curse of dimensionality): the optimal multi-step-ahead control has the widest gap between the optimal design and practically optimal design. It is rarely analytically or numerically feasible. For this reason, a lot of heuristic approximation techniques have been developed. Some of them are discussed in previous text.

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Indices

List of Terms II

List of Terms III

List of Terms IV

List of Terms VII

List of Terms VIII

List of Terms IX

List of Terms XIII

List of Terms XIV

List of Terms XVI

List of Examples I

List of Examples II

$\mathbf{1}$, `\1`	vector of units, 1
ℓ_X, `\Cv{X}`	length of vector X, 1
$\|X^\star\|$, `\S{X}`	cardinality of $X^\star$, 1
$\mathcal{D}i$, `\Di`	Dirrichlet rnd, 1
$\mathsf{D}\left(\mathsf{f}\|\|\mathsf{g}\right)$, `\D{\O{f}}{\O{g}}`	Kullback-Leibler divergence of f on g, 1
$\mathsf{E}\left[X\right]$,, `\Eu{X}`	unconditional expectation of X, 1
$\mathsf{E}\left[X\|\, Y\right]$, `\E{X}{Y}`	expectation of X conditioned on Y, 1
$\mathcal{G}i\mathcal{W}$, `\GiW`	Gauss-inverse-Wishart rnd, 1
$\mathcal{G}_t{=}\mathcal{G}_{A_t}$,, `\G{t}`	ignorance of the action A_t, 1
$\mathcal{K}_{t-1}{=}\mathcal{K}_{A_t}$, `\K{t-1}`	knowledge of the action A_t, 1
$\mathcal{X}$, `\M{X}`	mathcal font used for rare symbols, 1
$\mathcal{N}$, `\N`	normal (Gaussian) rnd, 1
X, `\O{X}`	mathsf font reserved for operators, 1
$\mathbf{X}$, `\R{X}`	bold mathematics font used for realisations, 1
$X^\star$, `\S{X}`	set of Xs, 1

Latex New Commands and Keywords, see mathematics II

${}^{a}X$, \U{a}{X}	left upper non-numerical index of X, 1
$\mathrm{d}X$, \d{X}	differential of X, \d{X}, 1
$\mathfrak{X}$, \fr{X}	$\mathfrak{mathfrakfonts}$, 1
$\mathsf{f}(X)$, \fu{X}	unconditional rnd of X, 1
$\mathsf{f}(X\vert Y)$, \f{X}{Y}	rnd of X conditioned on Y, 1
$X \in X^{\star}$, \is{X}	X in the set $X^{\star}$, 1
${}^{m:n}$, X\q{m:n}	upper index for sequence $X_m, \ldots, X_n$, 1
$\mathrm{supp}\,[X]$, \su{X}	support of X, 1
$X_{\star}$, \s{X}	subset of $X^{\star}$, 1
\begin{agr}..\end{agr} \label{agr:name}, 1	Agreement (definition), labelled
\begin{alg}..\end{alg}	Algorithm, labelled \label{alg:name}, 1
\begin{cor}..\end{cor}	Corollary, labelled \label{cor:name}, 1
\begin{exa}..\end{exa}	Example, labelled \label{exa:name}, 1
\begin{prb}..\end{prb}	Problem, labelled \label{prb:name}, 1

`\begin{pro}..\end{pro}`	Proposition (weaker theorem), referred `\label{pro:name}`, 1
`\begin{rem}..\end{rem}`	Remark, labelled `\label{rem:name}`, 1
`\begin{req}..\end{req}`	Requirement (condition), labelled `\label{req:name}`, 1
`\begin{thm}..\end{thm}`	Theorem, labelled `\label{thm:name}`, 1

Latex New Commands and Keywords, see mathematics IV